1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 13

Recall Given an $n \times n$ matrix A,

the **COFACTOR** of a_{ij} is $C_{i,j} = (-1)^{i+j} M_{ij}$,

where $M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$.

the MINOR of a_{ij}.

You can use cofactors & determinants to find inverses!

The adjoint matrix of A (adjugate matrix of A) is

$$\mathrm{adj}(A) = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \end{bmatrix}^T.$$

← the transpose of the matrix of cofactors of A

Fact If A is invertible, then $A^{-1} = \dfrac{1}{\det(A)} \mathrm{adj}(A)$

2x2 case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has cofactor matrix

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \quad \text{So } \operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

3x3 case No nice expression.

Example $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 1 \\ -3 & 2 & 2 \end{bmatrix}$. Find A^{-1}!

Solution Cofactor matrix: $\begin{bmatrix} 0 & -3 & 3 \\ 6 & 12 & -3 \\ -3 & -3 & 3 \end{bmatrix}$

Check!

$$C_{1,1} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2-2 = 0$$

$$C_{1,2} = -\begin{vmatrix} 0 & 1 \\ -3 & 2 \end{vmatrix} = -(0-(-3)) = -3 \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$C_{2,3} = -\begin{vmatrix} 3 & -1 \\ -3 & 2 \end{vmatrix} = -(6-3) = -3 \qquad \operatorname{adj}(A) = \begin{bmatrix} 0 & 6 & -3 \\ -3 & 12 & -3 \\ 3 & -3 & 3 \end{bmatrix}$$

$$\det(A) = 3(0) + (-1)(-3) + 2(3) = 9$$

$$\text{So } A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 6 & -3 \\ -3 & 12 & -3 \\ 3 & -3 & 3 \end{bmatrix}.$$

Check! $A^{-1}A = \frac{1}{9}\begin{bmatrix} 0 & 6 & -3 \\ -3 & 12 & -3 \\ 3 & -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 1 \\ -3 & 2 & 2 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I.$$

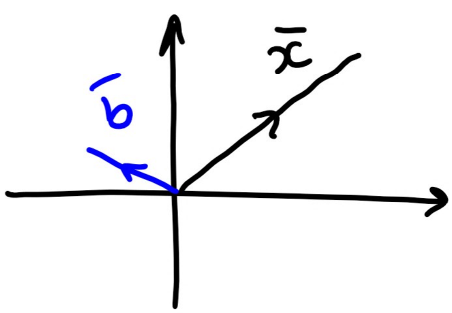
5.1 Eigenvalues & Eigenvectors

<u>Recall</u> We're writing systems of L.E.s as

$$\underset{m\times n}{A}\ \underset{n\times 1}{\bar{x}} = \underset{m\times 1}{\bar{b}}$$

If $m=n$, this says: multiply $\bar{x}$ by A (on left) & get vector $\bar{b}$

$\mathbb{R}^2$



A "distorts" space $\mathbb{R}^n$

(There are special A that rotate, project ... see Linear Algebra 2.)

Here given A $n\times n$, we're interested in finding those $\bar{x}\neq\bar{0}$ which <u>don't</u> get rotated by A (i.e. direction of $\bar{x}$ stays on same line through the origin as $\bar{x}$).

<u>Definition</u> If A is $n \times n$, we say $\bar{x} \neq \bar{0}$ is an <u>eigenvector of A</u> if $A\bar{x} = \lambda \bar{x}$ for some scalar λ which is called an <u>eigenvalue of A</u>. (We say "$\bar{x}$ corresponds to λ.")

<u>Examples</u>

$$\underbrace{\begin{bmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} -3-4+1 \\ 6-2+0 \\ 3-6+1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix} = \underbrace{-2}_{\lambda} \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{\bar{x}}$$

<u>Example</u>

$$\underbrace{\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \underbrace{5}_{\lambda} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\bar{x}}$$

<u>Example</u> Find <u>all</u> eigenvectors & eigenvalues of $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$.

<u>Solution</u> Want $\bar{x} \neq \bar{0}$ and λ with $A\bar{x} = \lambda \bar{x}$

$$\begin{bmatrix} x + 8y \\ 2x + y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

...?

In general, find eigenvalues λ of A first, then go back & find corresponding eigenvectors $\bar{x} \neq \bar{0}$.

Method to find λs.

We want λ, $\bar{x} \neq 0$ with $A\bar{x} = \lambda \bar{x}$.

$$= \underbrace{\lambda I}_{\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}} \bar{x}$$

So then $A\bar{x} - \lambda I \bar{x} = \bar{0}$

$$(A - \lambda I)\bar{x} = \bar{0}.$$

We get $\bar{x} \neq 0$ as solution if $A - \lambda I$ is NOT invertible.

We want λ with $\boxed{\det(A - \lambda I) = 0}$

Characteristic equation for A.

Does not involve $\bar{x}$. So solve for λ then go back & find corresponding $\bar{x}$.