1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 14

Yesterday The Eigenvalue/Eigenvector Problem

GOAL: Given A $n \times n$, find $\bar{x} \neq \bar{0}$ and λ with

$$A\bar{x} = \lambda \bar{x}$$

eigenvalue of A (scalar) ↑ ↖ eigenvector of A (vector!)

Find λ first by solving the characteristic equation of A:

$$\boxed{\det(A - \lambda I) = 0}$$

e.g. 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\det(A - \lambda I) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - \underbrace{(a+d)}_{= tr(A)}\lambda + \underbrace{(ad-bc)}_{= \det(A)}$$

In general $\det(A-\lambda I)$ is a polynomial in λ; we call that the characteristic polynomial of A.

Back to the Example: Find eigenvalues of $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$.

First set $\det(A - \lambda I) = 0$.

$$\det\left(\begin{bmatrix}1 & 8\\ 2 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = 0.$$

$$\begin{vmatrix}1-\lambda & 8\\ 2 & 1-\lambda\end{vmatrix} = 0$$

$$(1-\lambda)^2 - 16 = 0$$

$$\lambda^2 - 2\lambda + 1 - 16 = 0$$

$$\lambda^2 - 2\lambda - 15 = 0$$

$$(\lambda - 5)(\lambda + 3) = 0$$

So the eigenvalues of $A = \begin{bmatrix}1 & 8\\ 2 & 1\end{bmatrix}$ are $\lambda = 5, -3$.

To find the corresponding eigenvectors, take each λ in turn and solve for $\bar{x} \neq \bar{0}$ in $A\bar{x} = \lambda\bar{x}$ (or $(A - \lambda I)\bar{x} = \bar{0}$).

For our example: $A = \begin{bmatrix}1 & 8\\ 2 & 1\end{bmatrix}$

$\underline{\lambda = 5}$ $\quad (A - 5I)\bar{x} = \bar{0}$

$$\begin{bmatrix}\overset{-4}{\cancel{1-5}} & 8\\ 2 & \underset{-4}{\cancel{1-5}}\end{bmatrix}\bar{x} = \bar{0}$$

Augmented matrix : $\left[\begin{array}{cc|c} -4 & 8 & 0 \\ 2 & -4 & 0 \end{array}\right]$

$\xrightarrow[\text{to RREF}]{\text{Reduce}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x - 2y = 0$

$x = 2t$

$y = t$

So we get eigenvectors of form $\begin{pmatrix} 2t \\ t \end{pmatrix}$ for $t \neq 0$ ($t = 0$ gives $\bar{0}$ which is a solution to the equation $(A-5I)\bar{x} = \bar{0}$ but $\bar{0}$ is not allowed to be an eigenvector)

e.g. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$\lambda = -3$

$A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$

Look at $(A + 3I)\bar{x} = \bar{0}$.

$\begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix} \bar{x} = \bar{0}$

$\left[\begin{array}{cc|c} 4 & 8 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{Reduce}]{} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$

$y = t$

$\longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x + 2y = 0$

$x = -2t$

So eigenvectors corresponding to $\lambda = -3$ are of the form $\begin{pmatrix} -2t \\ t \end{pmatrix}$ e.g. $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

for $t \neq 0$

<u>Notice</u> We had ∞-many eigenvectors for each eigenvalue in the example.

In general, if $A\bar{x} = \lambda\bar{x}$, $\bar{x} \neq \bar{0}$, then

$$A(k\bar{x}) = k(A\bar{x}) = k(\lambda\bar{x}) = \lambda(k\bar{x})$$

for any scalar $k \neq 0$.

"Describing" eigenvectors

For a given eigenvalue λ, the collection of all eigenvectors corresponding to λ together with $\bar{x} = \bar{0}$ is the <u>eigenspace</u> of λ.

(i.e. all $\bar{x} \neq \bar{0}$ with $A\bar{x} = \lambda\bar{x}$) and $\bar{x} = \bar{0}$

Eigenspaces have a <u>basis</u> (which we'll define properly later) — it's a list of vectors in the eigenspace (not $= \bar{0}$)

(possibly only 1) $\swarrow$ 1 vector in the list, that is, a recipe for

$\underbrace{\text{producing}}$ all vectors $\bar{x}$ in the eigenspace

$\downarrow$

all eigenvectors are a linear combination of the vectors in the basis:

e.g. basis is $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$

linear combination of $\bar{x}_1, \ldots, \bar{x}_k$.

all eigenvectors look like $\boxed{t_1 \bar{x}_1 + t_2 \bar{x}_2 + \ldots + t_k \bar{x}_k}$

scalars

In our example, $\lambda = 5$ and -3. $\left(A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}\right)$

For $\lambda = -3$, eigenspace is

$$\left\{ \underbrace{\begin{pmatrix} -2t \\ t \end{pmatrix}}_{= \, t\begin{pmatrix} -2 \\ 1 \end{pmatrix}} \;\middle|\; t \in \mathbb{R} \right\}$$

any choice of $t \neq 0$ gives eigenvector for $\lambda = -3$

We would say $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ is a basis for this eigenspace.

(so is $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} -10 \\ 5 \end{pmatrix} \right\}$ or ...) $\leftarrow$ lots of choices, all have one vector

For $\lambda = 5$ we had eigenspace $\left\{ \begin{pmatrix} 2t \\ t \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

$t \neq 0$

Which has as a basis e.g. $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

$\vec{\text{eigenvector}}$

or $\left\{ \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 60 \\ 30 \end{pmatrix} \right\}$. etc.

(Again lots of choices, all with 1 vector.)

<u>Question</u> We know eigenvectors $\bar{x}$ <u>cannot</u> <u>ever</u> be $\bar{0}$.

But can eigenvalues λ be 0?

<u>Answer</u> <u>Yes</u>. This means there is an $\bar{x} \neq \bar{0}$ with $\boxed{A\bar{x} =} \, 0\bar{x} = \boxed{\bar{0}}$

It tells us A NOT invertible.

So now know for A $n \times n$,

A invertible $\Longleftrightarrow$ $\lambda = 0$ NOT an eigenvalue of A

$\Longleftrightarrow$ $\lambda = 0$ NOT a root of characteristic equation $\det(A - \lambda I) = 0$.

More parts to the "Big Theorem" on invertibility!

<u>Special case</u> Suppose $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots 0 & a_{nn} \end{bmatrix}$

upper triangular!

What are the eigenvalues of A?

Solve $0 = \det(A - \lambda I)$ also triangular

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots 0 & a_{nn}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots\cdots(a_{nn}-\lambda)$$

So the eigenvalues of A are $a_{11}, a_{22}, \dots, a_{nn}$

<u>Example</u> $\begin{bmatrix} 3 & 1 & 6 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ has eigenvalues $3, -2, 4$.

In general, eigenvalues of a triangular or diagonal matrix are the entries of main diagonal.