1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 15

Last Time More on the Eigenvalue/Eigenvector Problem

To find $\bar{x} \neq \bar{0}$ and λ with $A\bar{x} = \lambda\bar{x}$:

(1) First find all possible λ by solving $\det(A - \lambda I) = 0$.

(2) For each of the λ-values found in (1), solve $(A - \lambda I)\bar{x} = \bar{0}$.

Eigenspace of λ : All eigenvectors $\bar{x} \neq \bar{0}$ corresponding to λ + the zero vector $\bar{0}$

Basis for Eigenspace : Some list of eigenspace vectors that gets you all eigenspace vectors.

Last time $\xrightarrow{2\times 2}$ $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$ Eigenvalues $\lambda = 5$ and -3.

Eigenspace for $\boxed{\lambda = 5}$: has basis e.g. $\boxed{\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}}$ — all eigenvectors look like $t\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix}$

Eigenspace for $\boxed{\lambda = -3}$: has basis e.g. $\boxed{\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}}$ — all eigenvectors look like $t\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix}$.

In this example : 2 eigenvalues
1 basis vector for each eigenspace

Question: How many eigenvalues do we get?
" " eigenvectors in basis of each eigenspace?

Example $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & \cancel{3}4 & \cancel{4}3 \end{bmatrix}$. Find eigenvalues & eigenvectors.

Solution

$$0 = \det(A - \lambda I)$$

$$= \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & \cancel{3}4 & 3\cancel{4}-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ \cancel{3}4 & 3\cancel{4}-\lambda \end{vmatrix}$$

$$= (5-\lambda)\left((1-\lambda)(\overset{3}{\cancel{4}}-\lambda) - \overset{8}{\cancel{6}}\right)$$

$$= (5-\lambda)(\lambda^2 - 4\lambda - 5)$$

$$= (5-\lambda)(\lambda+1)(\lambda-5)$$

So $\lambda = -1, 5$ (5 is a repeated eigenvalue.)

$\underline{\lambda = -1}$ Solve $A\bar{x} = -\bar{x}$ i.e. $(A+I)\bar{x} = \bar{0}$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \xrightarrow[\cdots]{\text{Reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$x = 0$
$y = -z = -t$

eigenvectors: $t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $(t \neq 0)$

So basis for eigenspace: $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$.

$\underline{\lambda = 5}$ Now solve $A\bar{x} = 5\bar{x}$ i.e. $(A - 5I)\bar{x} = \bar{0}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right] \xrightarrow[\text{to RREF}]{\text{Reduce}} \left[\begin{array}{ccc|c} 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = t \qquad z = s$

$y - \frac{1}{2}z = 0$

$y = 1/2\, z = 1/2\, s$

Solutions : $\begin{pmatrix} t \\ s/2 \\ s \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}$

basis : $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$

Get 2 vectors in basis.

Recall: characteristic equation: $(\lambda+1)(\lambda-5)^2 = 0$

Was this ~~because~~ $\lambda = 5$ is a repeated root?

Alas, no:

<u>Example</u> $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $\det(A - \lambda I) = 0$ has roots $\lambda = 1$

So only 1 eigenvalue $\lambda = 1$.

Solve $A\bar{x} = \bar{x}$ i.e. $\underbrace{(A - I)\bar{x} = 0}$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow y = 0$

$x = t$

eigenvectors:

$t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad t \neq 0$

Basis : $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

Definitions (Multiplicity)

eigenvalue

Algebraic Multiplicity of λ

$\det(A - \lambda I)$ ← $AM(\lambda)$ is the # times λ appears as a root in characteristic polynomial

Geometric Multiplicity of λ

$GM(\lambda)$ = # eigenvectors in a basis for the eigenspace for λ

= # free variables in solutions to $(A - \lambda I)\bar{x} = \bar{0}$

= # non-pivot columns in RREF of $A - \lambda I$

$$1 \leq GM(\lambda) \leq AM(\lambda)$$

So for each eigenvalue λ you can have at most as many eigenvectors in a basis for its eigenspace as the # times λ appears as a root of the char. poly.

If all roots of characteristic polynomial are distinct (i.e. $AM(\lambda) = 1$ for every λ)

then $1 = AM(\lambda) \geq GM(\lambda) \geq 1$

$$\Rightarrow GM(\lambda) = 1.$$

$= AM(\lambda)$.

Say A is $n \times n$. We're interested in when sum over all λ of $GM(\lambda)$ (i.e. total # of basis vectors for all eigenspaces) $= n$.

5.2 Diagonalization

Let A $n \times n$, with eigenvalues $\lambda_1, \dots, \lambda_k$.

Say $GM(\lambda_1) = n_1$ = # eigenvectors in basis for eigenspace for λ_1

$GM(\lambda_2) = n_2$ = # " " " " λ_2

$\vdots$

$GM(\lambda_k) = n_k$ = # " " " " λ_k.

If $\underbrace{\overset{GM(\lambda_1)+GM(\lambda_2)+GM(\lambda_3)+\dots+GM(\lambda_k)}{n_1+n_2+n_3+\dots+n_k} = n}$, then ...

Notice $\boxed{n_i = GM(\lambda_i)}$

$n = AM(\lambda_1) + \dots + AM(\lambda_n)$

"degree of characteristic poly. = sum of # roots

This is a fact you can observe by thinking about the roots of the char. poly.

So this condition really says $AM(\lambda_i) = GM(\lambda_i)$ for every i

$AM(\lambda_i)$ = # times λ_i appears as root in char. poly. = # vectors in basis for λ_i-eigenspace $= GM(\lambda_i)$

... do the following : (You need n distinct eigenvectors to do this!)

Set $P = \begin{bmatrix} | & | & & | \\ P_1 & P_2 & \cdots & P_n \\ | & | & & | \end{bmatrix}$ where the columns $P_1, \dots, P_n$ are the

$n = n_1 + n_2 + \cdots + n_k$ many basis eigenvectors

Then <u>Fact</u> $D = P^{-1}AP$ is diagonal — in fact, d_{ii} = eigenvalue to which P_i corresponds.

<u>Example</u> Above : $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix} \leftarrow 3 \times 3$

eigenvalues :	$\lambda = -1$	5
# basis vectors in eigenspace	$1 = n_1$	$2 = n_2$
basis vectors	$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$.

$$n_1 + n_2 = 1 + 2 = 3 \ \checkmark$$

Set $P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1/2 \\ 1 & 0 & 1 \end{bmatrix}$. Then $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

<u>Check</u>!