1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 / C03) Lecture 16

Last Time ... we saw that if A is $n \times n$ and

A has n basis vectors across all of its eigenvalues,

[i.e. for each eigenvalue λ of A, $AM(\lambda) = GM(\lambda)$

$AM(\lambda)$: # times λ is a root of $\det(A - \lambda I)$ $GM(\lambda)$: # vectors in basis for eigenspace of λ]

then $\boxed{D = P^{-1}AP}$ $\leftarrow$ $P = \begin{bmatrix} | & & | \\ P_1 & \cdots & P_n \\ | & & | \end{bmatrix}$ with $AP_i = \lambda_i P_i$

diagonal: $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Similar Matrices

For A, B $n \times n$ we say B is similar to A if there is some invertible matrix P with

$$B = P^{-1}AP. \leftarrow$$

(**Notice**: if $B = P^{-1}AP$ then

$$PB = \cancel{PP^{-1}}AP$$

$$PBP^{-1} = A\cancel{PP^{-1}}$$ so if $Q = P^{-1}$

$$Q^{-1}BQ = A \Rightarrow A \text{ is similar to } B \text{ as well.})$$

Facts Suppose A & B are similar.

- $\det(A) = \det(B)$
- A invertible $\Leftrightarrow$ B invertible
- $\operatorname{tr}(A) = \operatorname{tr}(B)$
- A and B have the same characteristic polynomials $(\det(A-\lambda I) = \det(B-\lambda I))$.
- A and B have same eigenvalues
- If λ is an eigenvalue of A (and B), then # basis vectors in eigenspace of A corresponding to λ = # basis vectors in eigenspace of B corresponding to λ

Definition If A is similar to a diagonal matrix, call it D (i.e. there's an invertible P with $D = P^{-1}AP$), then we say that A is **diagonalizable**.

We say that P **diagonalizes** A.

So our Fact from last time can be reframed:

Fact If A $n \times n$ has n basis vectors for all its eigenspaces, then A is diagonalizable.

[$D = P^{-1} A P$]

diagonal entries eigenvalues of A

P has columns the corresponding basis vectors

The reverse is also true: Suppose A $n \times n$, P invertible, D diagonal with $\boxed{D = P^{-1} A P}$

Then $PD = AP$

$$\begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix} \qquad A \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix}$$

Columns of output on LHS are $\underline{d_{ii} p_i}$

Columns of output on RHS are $\underline{A p_i}$

$d_{ii} p_i = A p_i$

d_{ii} are eigenvalues of A with p_i corresponding eigenvectors

So in fact:

<u>Fact</u> A $n \times n$ diagonalizable $\Leftrightarrow$ A has n basis vectors across all eigenspaces

In particular, if A has a characteristic polynomial with n distinct roots, then A is diagonalizable.

To diagonalize a matrix A (if you can):

(1) Find all eigenvalues of A

(2) Find a basis of eigenvectors for each eigenspace of A

(3) Take stock: count the # basis vectors across all eigenspaces.

If total # basis vectors $= n$, where A is $n \times n$,

then:

Every eigenspace has ≥ 1 basis vector. So to check diagonalizability, just need to check the # basis vectors for repeated roots (should = # times root repeats)

(4) diagonalize A by writing $P = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix}$

where p_i are basis vectors for the eigenspaces.

(5) Then $P^{-1}AP$ is diagonal $= D$ where diagonal entries of D are the corresponding eigenvalues.

If A is triangular $n \times n$ then its eigenvalues are the diagonal entries. So if those diagonal entries are distinct then A is diagonalizable.

If those diagonal entries are not distinct, then you still need to check if there are enough basis vectors (A might be diagonalizable, or it might not).

The power of Diagonalizability

Suppose A $n \times n$ is diagonalizable, so there's invertible P & diagonal D with

$$\boxed{P^{-1} A P = D}$$

Then $A = PDP^{-1}$

$$\text{so } \quad A^k = \overbrace{(PD\cancel{P^{-1})(P}D\cancel{P^{-1})(} \ldots \cancel{)(P}DP^{-1})}^{k \text{ times}}$$

multiply A together k times

$$= P\overbrace{DD \ldots D}^{k \text{ times}}P^{-1}$$

$$= PD^kP^{-1}$$

$$D^k = \begin{bmatrix} d_{11}^k & & & 0 \\ & d_{22}^k & & \\ & & \ddots & \\ 0 & & & d_{nn}^k \end{bmatrix}$$

So PD^kP^{-1} is <u>much</u> easier to compute than A^k

<u>Example</u> Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$. Compute a general formula for A^k and then A^{15}.

<u>Solution</u> Try to diagonalize A:

Eigenvalues: $0 = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) - (-2)$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 2)(\lambda - 1)$$

So eigenvalues are $\lambda = 2$, $\lambda = 1$.

Eigenvectors: For $\lambda = 1$:

check!

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Basis: $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

For $\lambda = 2$: $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Basis: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Set $P = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}$. Then $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Now find $P^{-1} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$.

So $A^k = PD^kP^{-1} = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2^{k+1} - 1 & 1 - 2^k \\ 2^{k+1} - 2 & 2 - 2^k \end{bmatrix}.$$

So A^{15} $\longrightarrow$ plug in $k = 15$.

This example also worked through at the start of Lecture 17.