1ZC3 Engineering Math II-B (Linear Algebra I)

(WS 19 C03) Lecture 17

Last Time

DIAGONALIZATION

A diagonalizable : $P^{-1}AP = D$ — D diagonal, P invertible

(We might say "A is similar to a diagonal matrix".)

P : columns are basis vectors for eigenspaces of A

D : diagonal entries are eigenvalues of A

UP TO HERE FOR TEST 1

POWERS of diagonalizable matrices:

$$A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$$

Example from last time: $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$. Want A^k, A^{15}.

First find eigenvalues : $\lambda = 1, 2$.

Then basis vectors for eigenspaces : $\left\{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

Start with P. Choice : $\begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$

corresponds to $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ — corresponds to $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Find $P^{-1} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$

Now $A^k = \cancel{P^{-1}D^kP}\ PD^kP^{-1}$

$= \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \cancel{\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}} \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix} \cancel{\begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$

So $A^{15} = \begin{bmatrix} 2^{16}-1 & 1-2^{15} \\ 2^{16}-2 & 2-2^{15} \end{bmatrix}$

$= \begin{bmatrix} 65535 & -32767 \\ 65534 & -32766 \end{bmatrix}$.

$= \begin{bmatrix} 2^{k+1}-1 & 1-2^{k} \\ 2^{k+1}-2 & 2-2^{k} \end{bmatrix}$

Very sorry for this typo here. I really hope that it didn't confuse anyone too much. It appears correctly at the end of Lecture 16. Please make sure you remember the formulas correctly: $A = PDP^{-1}$; $D = P^{-1}AP$. $A^k = PD^kP^{-1}$. It is possible, even after a long time, to get this wrong! But you'll be OK if you are aware in your studying that it is an issue.

5.4 Differential Equations (DEs)

We're interested in this type: $\boxed{y' = ky}$

$(y = f(t))$ $\quad$ (k real #)

$$\int \frac{y'}{y} = \int k \Rightarrow \ln|y| = kt + C$$

$$\Rightarrow |y| = e^{kt+C}$$

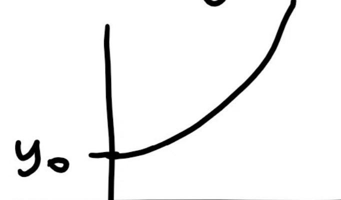
$$\Rightarrow y = \pm e^{kt+C}$$

$$\Rightarrow \boxed{y = y_0 e^{kt}} \quad (y_0 = \pm e^C)$$

General Solution

Notice $y(0) = y_0 e^{k \cdot 0} = y_0 e^0 = y_0 \cdot 1 = y_0$

Exponential Growth/Decay:



$k > 0$

$k < 0$

- Population Growth
- Wealth (compound interest)
- Spread of Infection
- Radioactive Decay
- Metabolic disintegration

Systems of DEs

e.g. Populations interact & rate of change depends on total size of each population.

(1) Zombie apocalypse, population of humans $h(t)$ & pop. of zombies $z(t)$, could be modelled by

$$h' = 3h - z$$
$$z' = 2h$$

Sometimes more complicated single DEs can be written as a system of DEs of our type:

(2) Suppose $y''' + 3y'' - 2y' + 6y = 0$.
(weighted sum of individual derivatives.)

Rename $y_1 = y$, $\boxed{y_2 = y_1'}$ $(= y')$, $y_3 = y_2'$ $(= y'')$

Rewriting: $y_3' + 3y_3 - 2y_2 + 6y_1 = 0$

The equations in red boxes form a system of DEs.

$$\Rightarrow \boxed{y_3' = -6y_1 + 2y_2 - 3y_3}$$

To solve systems of DEs like this:

1st big idea: Write a system as:

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \bar{y}' = A\bar{y} = A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

A is $n \times n$ matrix of coefficients

Examples above:

① (Zombies) $\bar{y} = \begin{bmatrix} h \\ z \end{bmatrix}$ & $\bar{y}' = \begin{bmatrix} h' \\ z' \end{bmatrix} = \begin{bmatrix} 3h - z \\ 2h \end{bmatrix}$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \bar{y}$$

② (DE) $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\bar{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 2 & -3 \end{bmatrix} \bar{y}.$

Suppose we have system of DEs $\bar{y}' = A\bar{y}$.

If A diagonal then this is

$$\left.\begin{aligned} y_1' &= a_{11} y_1 \\ y_2' &= a_{22} y_2 \\ &\vdots \\ y_n' &= a_{nn} y_n \end{aligned}\right\} \text{uncoupled equations — can solve}$$

→ Solution: $y_1 = y_{1,0} e^{a_{11} t}$
$\vdots$
$y_n = y_{n,0} e^{a_{nn} t}$

(Follows our General Solution above.)

$y_{i,0} = y_i(0)$

<u>2nd big idea</u> If A is diagonalizable, we can do something very similar.

$A = PDP^{-1}$ with D diagonal

Now if $\bar{y}' = A\bar{y} = (PDP^{-1})\bar{y}$

$$\Rightarrow P^{-1}\bar{y}' = D(P^{-1}\bar{y})$$

Make a change of variables: set $\bar{u} = P^{-1}\bar{y}$.

Then $\bar{u}' = D\bar{u}$.

We know solution from before: since D diagonal

$$u_i = u_{i,0} e^{d_{ii} t} \qquad u_{i,0} = u_i(0)$$

Initial value of u_i, which we can get from $\bar{u}(0) = (P^{-1}\bar{y})(0)$ i.e. $\begin{bmatrix} u_{1,0} \\ \vdots \\ u_{n,0} \end{bmatrix} = P^{-1} \begin{bmatrix} y_{1,0} \\ \vdots \\ y_{n,0} \end{bmatrix}$

So we get $\boxed{\bar{y} = P\bar{u}}$

Recall: column P_i of P is eigenvector corresponding to d_{ii}

So $\bar{y} = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} \begin{bmatrix} u_{1,0}\, e^{d_{11}t} \\ \vdots \\ u_{n,0}\, e^{d_{nn}t} \end{bmatrix}$

$$\boxed{\bar{y} = u_{1,0}\, \underbrace{P_1}_{\text{eigenvector}}\, e^{\overbrace{d_{11}}^{\text{eigenvalue}}t} + \cdots + u_{n,0}\, P_n\, e^{d_{nn}t}}$$

Example ① solved: $\bar{y}' = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \bar{y}$

Initially 1000 humans
75 zombies. So $\bar{y}_0 = \begin{bmatrix} 1000 \\ 75 \end{bmatrix}$

From earlier example:

$A = PDP^{-1}$

$$= \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$$

p_1, p_2; d_{11}, d_{22}; P^{-1}

$$\bar{u}_0 = P^{-1} \begin{bmatrix} 1000 \\ 75 \end{bmatrix} = \begin{bmatrix} -1850 \\ 1925 \end{bmatrix}$$

$$\bar{y} = u_{1,0} P_1 e^{d_{11}t} + u_{2,0} P_2 e^{d_{22}t}$$

$$= -1850 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{t} + 1925 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$