1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 18

Today Complex Numbers

See Chapter 10 of 9th Edition of Anton & Rorres textbook.

($\longrightarrow$ find on Course Website.)

Example Find the eigenvalues of $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$.

Solve $0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix}$

$$= (2-\lambda)(-2-\lambda) - (-5)$$

$$= -4 + \cancel{2\lambda} - \cancel{2\lambda} + \lambda^2 + 5$$

$$= \lambda^2 + 1.$$

Want λ with $\lambda^2 = -1$?!

Isn't $x^2 \geq 0$ for any x? Yes if x is real.

We need to make new numbers!

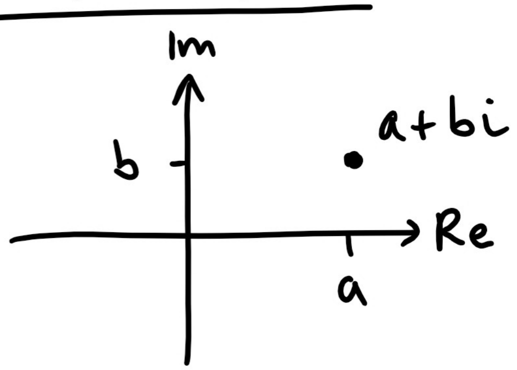
Definitions The **imaginary unit** i is the "number" satisfying $\boxed{i^2 = -1}$.

Complex (AKA **imaginary**) **numbers** are "numbers" of the form $z = a + bi$ where a, b are real numbers, called

a: real part of z

b: imaginary part of z

We can visualize complex #s $z = a+bi$ as vectors (a, b) in the **Complex plane** (AKA **Argand diagram**)



with real & imaginary axes.

Example

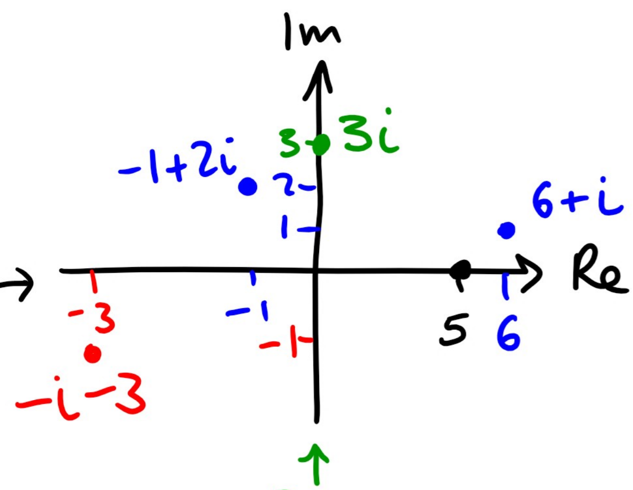
$6 + i$

$-1 + 2i$

$-i - 3$

$3i$ — called **purely imaginary** (no real part)

5 — called **real**! (no imaginary part)



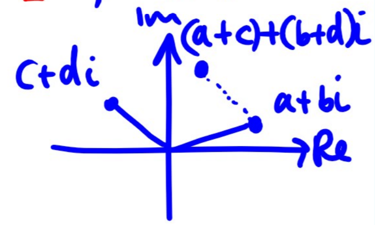
Operations on Complex Numbers

- Addition (/Subtraction) $\quad (a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$

Vectors: $(a,b) \pm (c,d) = (a \pm c,\ b \pm d)$

$= a + bi + c + di$

$= a + c + bi + di$



Example $\quad (1-i) + (6-3i) = 7 - 4i.$

- Multiplication $\quad (a+bi)(c+di) = ac + adi + bci \boxed{+ bd i^2}$ $= -bd$

↳ No good vector analog (unless $b=0$ or $d=0$) — at least, not in the way we write complex #s as vectors so far — more next time!)

$= (ac - bd) + (ad + bc)i$

Example $\quad (1-i)(6+3i) = 6 - 6i + 3i \boxed{-3i^2}$ $\quad (-3)(-1) = +3$

$= 9 - 3i.$

Example $\quad \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = \ldots$

$\left(\frac{1}{\sqrt{2}}\right)(1+i)\left(\frac{1}{\sqrt{2}}\right)(1+i) = \frac{1}{2}(1+i)(1+i)$

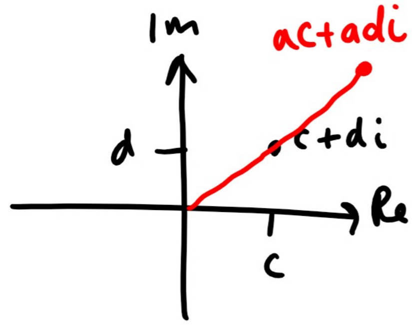
$= \frac{1}{2}\left(1 + i + i \boxed{+ i^2}\right) = \frac{2i}{2} = i$ $\quad -1$

Does this mean that $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ is a square root of i? Yes! (More on this tomorrow.)

→ Actually Lecture 20.

Above if $b = 0$, so we have $\underset{\substack{\uparrow \\ \text{real \#}}}{a}(c+di) = ac + adi$

"Scalar multiplication" by a real # (a)



Now we can add/subtract/multiply complex #s, we can work with vectors & matrices whose entries are complex #s — matrix operations carry over.

<u>Example</u>

$$\underbrace{(1+3i)}_{k}\underbrace{\begin{bmatrix} 1 & -i \\ -2+i & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1+3i & (1+3i)(-i) \\ (1+3i)(-2+i) & 0 \end{bmatrix}$$

This is scalar multiplication kA, where both k is a complex # and A is a matrix with complex entries.

$$= \begin{bmatrix} 1+3i & -i-3i^2 \\ -2-6i+i\underbrace{+3i^2}_{-3} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3i & 3-i \\ -5-5i & 0 \end{bmatrix}.$$

<u>Example</u> Above we showed that $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

had eigenvalues λ satisfying $\lambda^2+1=0$

i.e. $\lambda = \pm i$

To find the eigenvectors, procedure is <u>exactly</u> the same as you know e.g. solve $(A - iI)\bar{x} = \bar{0}$.

----> for example we could find:

$$\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2(2+i) - 5 \\ 2+i \quad -2 \end{bmatrix} = \begin{bmatrix} -1+2i \\ i \end{bmatrix}$$

$$= i\begin{bmatrix} 2+i \\ 1 \end{bmatrix}.$$

(So $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = i$.)

- <u>Division</u> of Complex #s

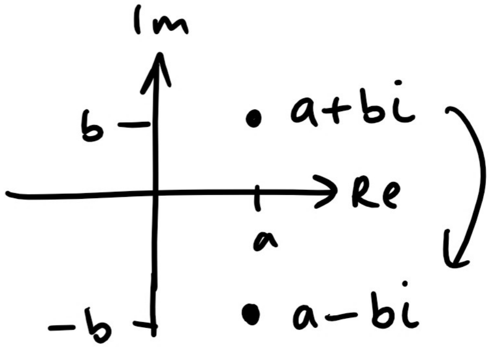
Given $w = a+bi$ & $z = c+di$,

What is $\frac{w}{z} = \frac{a+bi}{c+di}$? $= x+yi$

The answer is $x+yi$ where $a+bi = (x+yi)(c+di)$

To answer this question in a nice way, we need:

<u>Definitions</u> The <u>complex conjugate</u> of $z = a+bi$ is $\bar{z} = a - bi$



Mirror in real axis

<u>Examples</u> $z = 6+5i$; $\bar{z} = 6-5i$

$z = -3-2i$; $\bar{z} = -3+2i$

$z = 4i - 1$; $\bar{z} = -4i - 1$

$z = 3$; $\bar{z} = 3$

$z = -i$; $\bar{z} = i$

T.B.C.