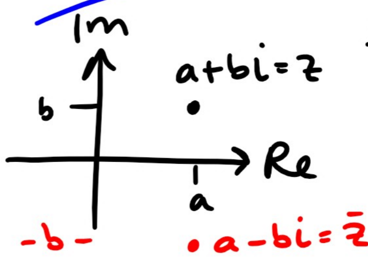
1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 19

Yesterday — Complex Numbers



$$\underbrace{z = a}_{\text{Real part}} + \underbrace{b}_{\text{Imaginary part}}i \quad \text{, where } \boxed{i^2 = -1}.$$

Addition / Subtraction: $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$.

Multiplication: $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$.

Division: today.

The complex conjugate of $z = a+bi$ is $\bar{z} = a - bi$.

Useful facts about complex conjugates

$$z + \bar{z} = (a+bi) + (a-bi) = 2a \leftarrow \text{always real.}$$

$$z - \bar{z} = (a+bi) - (a-bi) = 2bi \leftarrow \text{always purely imaginary}$$

$$z\bar{z} = (a+bi)(a-bi) = a^2 + \cancel{abi} - \cancel{bai} \underbrace{\boxed{-b^2 i^2}}_{+b^2}$$

$$= a^2 + b^2 \leftarrow \text{always real \& non-negative}$$

The **modulus/absolute value** of a complex # $z = a+bi$

is $|z| = \sqrt{z\bar{z}} \quad \left(= \sqrt{a^2+b^2} \right)$

(So above we showed $z\bar{z} = |z|^2$.)

Notice : If $z = a \in \mathbb{R}$ (i.e. $b=0$), then

$$|z| = \sqrt{a^2} = |a|$$

(so this agrees with the definition of absolute value for real #s.)

Back to : $\frac{w}{z} = \frac{a+bi}{c+di}$?? $(= x + yi)$

Trick that you should use :

multiply $\frac{w}{z}$ by $\frac{\bar{z}}{\bar{z}} = 1$:

$$\frac{w}{z} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{\underbrace{(c+di)(c-di)}_{z\bar{z} = |z|^2 = c^2+d^2}} = \overbrace{\boxed{\frac{(ac+bd)}{c^2+d^2}}}^{x} + \overbrace{\boxed{\frac{(bc-ad)}{c^2+d^2}i}}^{y}$$

This works as long as $|z|^2 \neq 0$ i.e.

$c^2+d^2 \neq 0$ i.e.

both $c \neq 0$ & $d \neq 0$ i.e. $z \neq 0$

Example Find $\frac{6+7i}{-3+2i}$.

Method above: $\hookrightarrow = \frac{(6+7i)(-3-2i)}{(-3+2i)(-3-2i)} = \frac{-18-21i-12i-14i^2}{(-3)^2+2^2}$

$z \longrightarrow \bar{z}$

$$= \frac{-4-33i}{13}$$

$$= -\frac{4}{13} - \frac{33}{13}i.$$

Special case of Division:

$$z^{-1} = \frac{1}{z} = \frac{1}{z}\left(\frac{\bar{z}}{\bar{z}}\right) = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$ (as long as $z \neq 0$)

For example above, another approach $\left(\frac{6+7i}{-3+2i}\right)$:

$$\frac{6+7i}{-3+2i} = (6+7i)\left(\frac{1}{-3+2i}\right).$$ So first find $\frac{1}{\underbrace{-3+2i}_{z}}$

$$= \frac{\overbrace{-3-2i}^{\bar{z}}}{\underbrace{(-3)^2+2^2}_{|z|^2}} = -\frac{3}{13} - \frac{2}{13}i.$$

$$= (6+7i)\left(\frac{-3}{13} - \frac{2}{13}i\right)$$

$$= -\frac{18}{13} - \frac{12}{13}i - \frac{21}{13}i - \frac{14}{13}i^2 = -\frac{4}{13} - \frac{33}{13}i.$$

More Facts about Complex Conjugates

(1) $\bar{\bar{z}} = z$

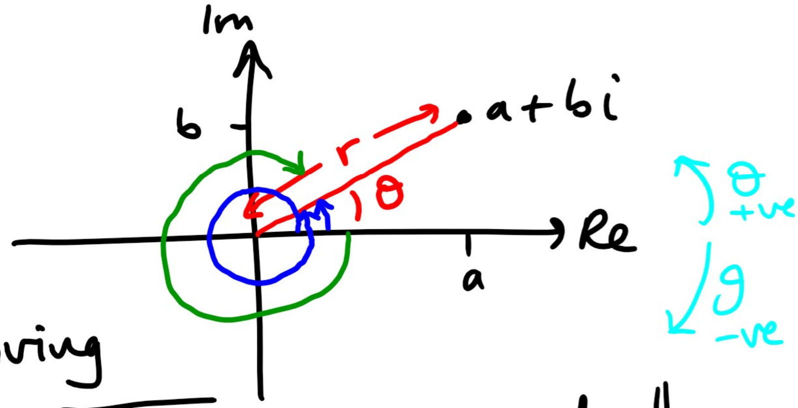
(2) $\overline{w \pm z} = \bar{w} \pm \bar{z}$

(3) $\overline{wz} = \bar{w}\,\bar{z}$

(4) $\overline{\left(\frac{w}{z}\right)} = \frac{\bar{w}}{\bar{z}}$.

Polar Form of complex Numbers

Represent $z = a+bi$



We can plot z by giving

a radius $r = |z| = \sqrt{a^2+b^2}$ ← a real #

& an angle θ s.t. $a = r\cos\theta$
$b = r\sin\theta$

relative to (positive) Real axis

i.e. there are infinitely many θ that will plot the same complex # z
(Find one, θ, and all the others are $\theta + 2\pi k$ for an integer k.)

θ : argument for z, $\arg(z)$

By convention, the $\arg(z)$ lying in $(-\pi, \pi]$ is called the <u>principal argument</u> for z, $\mathrm{Arg}(z)$
↑ capital A

This gives us "the" <u>polar form</u> of $z = a+bi$

$$= r\cos\theta + (r\sin\theta)i$$
$$= r(\cos\theta + i\sin\theta)$$

<u>Example</u> Find the polar forms of (i) $z = -3-\sqrt{3}i$
(ii) $z = 2i$

<u>Solutions</u> (i) $r = |z| = \sqrt{(-3)^2+(-\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12}$
$= 2\sqrt{3}$.

Need θ s.t. $-3 = r\cos\theta = 2\sqrt{3}\cos\theta$

$\Rightarrow \cos\theta = \dfrac{-3}{2\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$ e.g. $\theta = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \ldots$

$-\dfrac{5\pi}{6}, -\dfrac{7\pi}{6}, \ldots$

also s.t. $-\sqrt{3} = r\sin\theta = 2\sqrt{3}\sin\theta$

$\Rightarrow \sin\theta = \dfrac{-\sqrt{3}}{2\sqrt{3}} = -\dfrac{1}{2}$ e.g. $\theta = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}, \ldots$

$\dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \ldots$

Notice: it's enough to find one value for θ that is on both lists, as all other arguments are $\theta + 2\pi k$ for integers k.

say $\theta = -\dfrac{5\pi}{6} + 2\pi k$, k integer.

$\mathrm{Arg}(-3-\sqrt{3}i)$

So $z = -3-\sqrt{3}i = 2\sqrt{3}\left(\cos\left(-\dfrac{5\pi}{6}\right) + i\sin\left(-\dfrac{5\pi}{6}\right)\right)$

or any other argument

(ii) $z = 2i$

$r = 2$

θ satisfies $0 = 2\cos\theta \Rightarrow \cos\theta = 0$

$2 = 2\sin\theta \Rightarrow \sin\theta = 1$

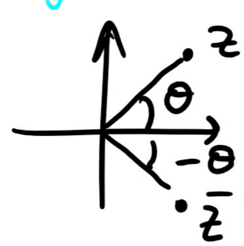
solutions: $\theta = \dfrac{\pi}{2} + 2\pi k$, k integer.

So $2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$, say.

Notice If $z = r(\cos\theta + i\sin\theta)$

then $\bar{z} = r(\cos\theta - i\sin\theta)$

$= r(\cos(-\theta) + i\sin(-\theta))$



i.e. arguments for $\bar{z}$ are the negative of the args for z.

Polar form $z = r\underbrace{(\cos\theta + i\sin\theta)}$

Using Maclaurin series, can show $\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$

Need to know this; do not need to know why.

So $z = re^{i\theta}$

& multiplication & division get a lot easier:

Say $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} = \overbrace{r_1 r_2}^{\text{Multiply moduli}}(\cos\overbrace{(\theta_1+\theta_2)}^{\text{add args}} + i\sin(\theta_1+\theta_2))$

$|z_1 z_2| = |z_1||z_2|$

$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \underbrace{\frac{r_1}{r_2}}_{\text{divide moduli}} e^{i\underbrace{(\theta_1-\theta_2)}_{\text{subtract arguments.}}}$

↑ If $z_2 \neq 0$.

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)).$