1ZC3 Engineering Math II-B (Linear Algebra I)

(WS 19 / C03) Lecture 2

Yesterday: Systems of Linear Equations

m equations in n variables

$$\left[\begin{array}{lllllll}
① & a_{11}x_1 & + a_{12}x_2 & + & \cdots & + a_{1n}x_n & = b_1 \\
② & a_{21}x_1 & + a_{22}x_2 & + & \cdots & + a_{2n}x_n & = b_2 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots \\
ⓜ & a_{m1}x_1 & + a_{m2}x_2 & + & \cdots & + a_{mn}x_n & = b_m
\end{array}\right.$$

a_{ij} = coefficient in equation # i of variable x_j

A solution is an n-tuple solving all m equations at once.

We can represent a system of L.E.S using a matrix (rectangular array of numbers):

the augmented matrix of the above system is

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & & & & \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]$$

line optimal

Back to lines: yesterday we saw

$$\left\{\begin{array}{l} x + 3y = 6 \\ 2x - y = 1 \end{array}\right\}$$ had 1 solution.

Example $\left\{\begin{array}{l} x + y = 5 \\ 3x + 3y = 15 \end{array}\right\}$

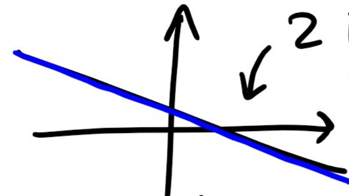
$\downarrow$

our earlier strategy:

$$3x + 3y - 3(x+y) = 15 - 3.5$$

$$0 = 0$$

– no extra information from having a second equation



every point on line is a solution i.e. $\underline{\underline{\infty}}$ - many Solutions.

$\downarrow$

(x, y) with $y = 5 - x$

We can write this using a parameter, say t :

Solutions are $(t, 5-t)$ for any value of t

e.g. $(0, 5)$ or $(\pi, 5-\pi)$ or $(-\sqrt{3}, 5+\sqrt{3})$

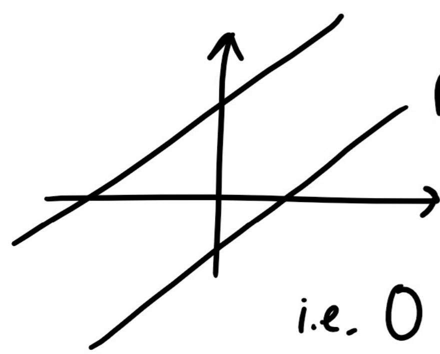
Example $\left\{\begin{array}{l} x - 2y = -10 \\ 2x - 4y = 6 \end{array}\right\}$

$\downarrow$

$$2x - 4y - 2(x - 2y) = 6 - 2(-10)$$

$$0 = 26$$

(nonsense!)



i.e. $\underline{\underline{0}}$ solutions

In fact, these 3 cases cover the only possibilities for <u>any</u> system of L.E.s : we will show in this course that every system has either

– 0 solutions $\longrightarrow$ inconsistent system

– 1 solution

– ∞- many solutions $\Big\}\rightarrow$ consistent system

Now consider the following example along with matrix notation:

Example Solve

Augmented matrix:

① $x - 3y + z = 1$

② $-2x + y - 2z = 1$

③ $2x - y - z = 1$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 2 & -1 & -1 & 1 \end{bmatrix}$$

③ → ② + ③ to elim. x & y from ③:

$R_3 \to R_2 + R_3$

① $x - 3y + z = 1$

② $-2x + y - 2z = 1$

④ $-3z = 2$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

④ → $-\frac{1}{3}$④ to solve for z:

$R_3 \to -\frac{1}{3} R_3$

① $x - 3y + z = 1$

② $-2x + y - 2z = 1$

⑤ $z = -\frac{2}{3}$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

② → 2·① + ② to elim. x & z from ②:

$R_2 \to 2R_1 + R_2$

① $x - 3y + z = 1$

⑥ $-5y = 3$

⑤ $z = -\frac{2}{3}$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & -5 & 0 & 3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

⑥ $\to -\frac{1}{5}$⑥ to solve for y: $\qquad R_2 \to -\frac{1}{5}R_2$

① $x - 3y + z = 1$

⑦ $y = -3/5$

⑤ $z = -\frac{2}{3}$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

① $\to$ ① $+ 3$⑦ $-$ ⑤ to solve for x: $\qquad R_1 \to R_1 + 3R_2 - R_3$

⑧ $x = 1 - \frac{9}{5} + \frac{2}{3} = -\frac{2}{15}$

⑦ $y = -3/5$

⑤ $z = -2/3$

$$A = \begin{bmatrix} 1 & 0 & 0 & -2/15 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

The unique solution is $(x, y, z) = \left(-\frac{2}{15}, -\frac{3}{5}, -\frac{2}{3}\right)$.

Return to earlier examples:

$x + y = 5$

$3x + 3y = 15$

$$\begin{bmatrix} 1 & 1 & 5 \\ 3 & 3 & 15 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = B$$

$x - 2y = -10$

$2x - 4y = 6$

$$\begin{bmatrix} 1 & -2 & -10 \\ 2 & -4 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 1 & -2 & -10 \\ 0 & 0 & 26 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{26}R_2} \begin{bmatrix} 1 & -2 & -10 \\ 0 & 0 & 1 \end{bmatrix} = C$$

Important Properties All of A, B, C satisfy:

(1) In every row, the "leading entry" i.e. the left-most non-zero entry is a 1 (if there is a leading entry)

(2) All zero rows are at the bottom.

(3) Every "leading 1" (as in (1)) in a non-zero row is to the right of all the leading 1s in rows higher up.

A & B also satisfy:

(4) A leading 1 is the only non-zero entry in its column.

Definition (REF) A matrix satisfying (1)-(3) is in row echelon form.

(RREF) A matrix satisfying (1)-(4) is in reduced row echelon form.

A pivot column is a column with a leading 1.

A free variable is a variable corresponding to a non-pivot column.

Examples

$$\begin{bmatrix} 1 & 2 & 1 & -5 & 0 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 6 \end{bmatrix}$$ ✗ REF ✗ RREF

$$\begin{bmatrix} 1 & 2 & 1 & -5 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 6 \end{bmatrix}$$ ✓ REF ✗ RREF