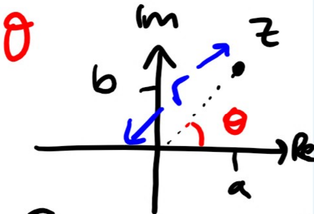
1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

Last Time

Complex Numbers: Polar Form $\binom{\text{WS } 19}{\text{C03}}$ Lecture 20



$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- radius $r = |z|$
- argument any θ satisfying $a = r\cos\theta$, $b = r\sin\theta$. [$\theta \in (-\pi, \pi]$ — principal]

Multiplication/Division:

$$(r_1(\cos\theta_1 + i\sin\theta_1))(r_2(\cos\theta_2 + i\sin\theta_2)) = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

[To multiply : multiply radii, add arguments.]

$$(r_1(\cos\theta_1 + i\sin\theta_1))/(r_2(\cos\theta_2 + i\sin\theta_2)) = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

[To divide : divide radii, subtract arguments.]

Example Find (i) $(2i)(-3-\sqrt{3}i)$ in polar form.

(ii) $2i / (-3-\sqrt{3}i)$

Last time we found $2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$$-3-\sqrt{3}i = 2\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right).$$

Solution (i) $(2i)(-3-\sqrt{3}i) = 2(2\sqrt{3})\left(\cos\left(\frac{\pi}{2}+\left(-\frac{5\pi}{6}\right)\right) + i\sin\left(\frac{\pi}{2}+\left(-\frac{5\pi}{6}\right)\right)\right)$

$$= 4\sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right).$$

(ii) $\dfrac{2i}{-3-\sqrt{3}i} = \dfrac{2}{2\sqrt{3}}\left(\cos\left(\frac{\pi}{2}-\left(-\frac{5\pi}{6}\right)\right) + i\sin\left(\frac{\pi}{2}-\left(-\frac{5\pi}{6}\right)\right)\right)$

$$= \frac{1}{\sqrt{3}}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) \quad \text{OK.}$$

$$= \frac{1}{\sqrt{3}}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

↑ principal argument.

<u>Special Case</u> $\quad z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$(r(\cos\theta + i\sin\theta))^n$ $\quad = r^n \boxed{(\cos(n\theta) + i\sin(n\theta))}$

$= r^n \boxed{(\cos\theta + i\sin\theta)^n}$ $\longleftrightarrow$ $=$ De Moivre's Rule.

<u>Finding roots</u> $\quad$ Let $z_1 = r_1 e^{i\theta_1}$.

What is $z_1^{1/n}$? It is some $z_2 = r_2 e^{i\theta_2}$ with $z_2^n = z_1$.

Goal: find possible values for r_2 & θ_2.

$$z_2^n = z_1$$

$$(r_2 e^{i\theta_2})^n = r_1 e^{i\theta_1} \quad \Rightarrow r_2^n = r_1$$

$$" \; r_2^n e^{in\theta_2} \quad \Rightarrow r_2 = \sqrt[n]{r_1} \quad \text{(all real)}$$

For θ_2: we need to find all θ_2 with

$n\theta_2$ some argument of z_1

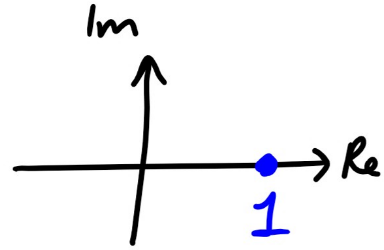
We can write all arguments of z_1 as $\alpha_1 + 2\pi k$, where α_1 is $\operatorname{Arg}(z_1)$ (principal arg. of z_1) &

k integer

i.e. $n\theta_2 = \alpha_1 + 2\pi k$ for some integer k

$$\boxed{\theta_2 = \frac{\alpha_1}{n} + \frac{2\pi k}{n}}$$ for some integer k

Enough to take $\boxed{k = 0, 1, \ldots, n-1}$ to get all possible values for θ_2 (after that, list repeats).



__Example__ Find cube roots of 1.

$\hookrightarrow n = 3$

__Solution__ 1 has radius $|1| = 1$

& principal argument is $0 = \alpha_1$

So our cube roots z have radius $= \sqrt[3]{1} = 1$

& arguments $= \dfrac{\alpha_1}{3} + \dfrac{2\pi k}{3}$ for $k = 0, 1, 2$.

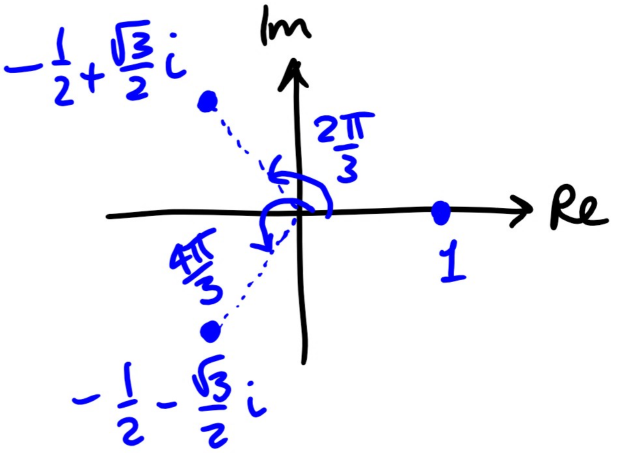
$$= \frac{0}{3} + \frac{2\pi k}{3} = \frac{2\pi k}{3} \quad \text{for } k = 0, 1, 2.$$

So cube roots of 1 are:

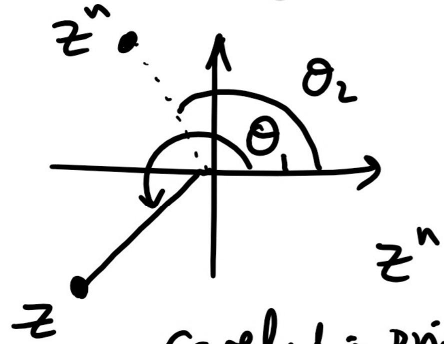
$(k=0)$: $1(\cos(0) + i\sin(0)) = 1$

$(k=1)$: $1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$(k=2)$: $1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.



Cautionary Tale:



careful: principal argument of z^n could be smaller than the argument of z.

<u>Example</u> Find the fourth $\leftarrow n=4$ roots of $-3-\sqrt{3}i$.

<u>Solution</u> Recall: $-3-\sqrt{3}i = 2\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

<u>Rules</u>: radius of roots $= |-3-\sqrt{3}i|^{1/4} = (2\sqrt{3})^{1/4}$

$= 2^{1/4}3^{1/8}$.

args. of roots: $\frac{\alpha_1}{n} + \frac{2\pi k}{n}$ for $k = 0, \ldots, n-1$

$= \left(-\frac{5\pi}{6}\right)/4 + \frac{\cancel{2}\pi k}{\cancel{4}\,2}$ for $k = 0, 1, 2, 3$.

So $(-3-\sqrt{3}i)^{1/4} = 2^{1/4}3^{1/8}(\cos\theta + i\sin\theta)$ where

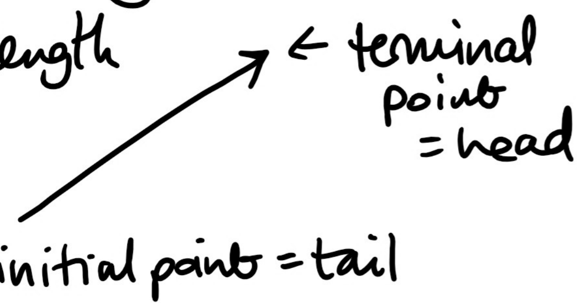
θ is one of: $-\frac{5\pi}{24}$, $-\frac{5\pi}{24}+\frac{\pi}{2}$, $-\frac{5\pi}{24}+\pi$,

$-\frac{5\pi}{24}+\frac{3\pi}{2}$.

3.1 Vectors in 2-space, 3-space, n-space

Back to the real world (for now...)

Vectors are
- matrices: $1 \times n$ (row vectors) & $m \times 1$ (column vectors)
- in $\mathbb{R}^2$, $\mathbb{R}^3$ we have line segments with a certain magnitude/length & direction



↳ placement is irrelevant i.e.

2 vectors are the same if length & direction are same.

↳ so we encode each vector (line segment) by placing its tail at the origin and naming the vector by the (x, y) or (x, y, z) location of the head.

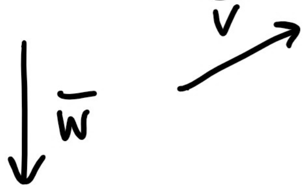
This is the row vector $[x \quad y]$ or column vector $\begin{bmatrix} x \\ y \end{bmatrix}$

also the 2-tuple (x, y) with <u>components</u> x & y.

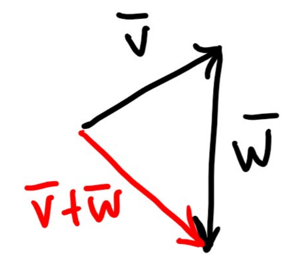
So $\mathbb{R}^2$ (called 2-space) is just all 2-tuples
& $\mathbb{R}^3$ (called 3-space) is just all 3-tuples

Geometry & Arithmetic agree in the obvious way!

- can <u>add</u> vectors:

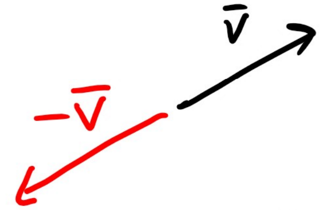


we get $\bar{v}+\bar{w}$ by:



e.g. $(2,3)+(0,-5)$
$= (2,-2)$.

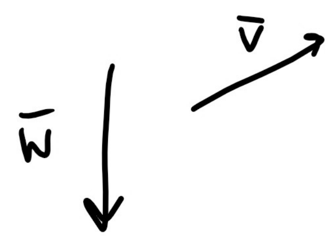
- <u>negate</u> vectors:



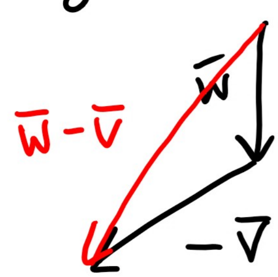
$-\bar{v}$ is the vector with magnitude same as $\bar{v}$ & opposite direction

e.g. $-(2,3) = (-2,-3)$.

- <u>subtract</u> vectors

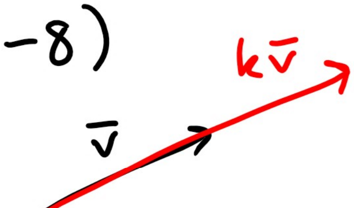


get $\bar{w}-\bar{v}$:



e.g. $(0,-5)-(2,3)$
$= (-2,-8)$

- <u>scaling</u> vectors



$k\bar{v}$ has magnitude $k \times$ magnitude of $\bar{v}$
& same direction as $\bar{v}$

if $k > 0$, opp. direction if $k < 0$

e.g. $2(2,3) = (4,6)$.

The point is, we can talk about all these geometric operations in terms of component-wise arithmetic ops.

<u>Goal</u>: translate geometric ideas to $\mathbb{R}^n$ (n-space).

→ using the fact that we can describe geometric ops from $\mathbb{R}^2$ & $\mathbb{R}^3$ in terms of arithmetic operations, even though we cannot visualize vectors as line segments in $\mathbb{R}^n$!