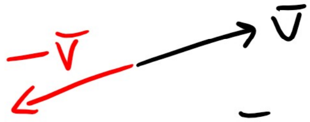
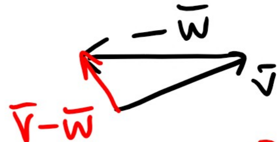
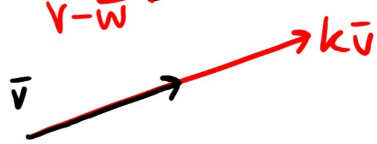
1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 21

Last Time

Vectors in 2-space / 3-space, ...

We equate line segments with 2-tuples / 3-tuples:

Line Segments	Component-wise Arithmetic
• ADD 	• e.g. $(5,-3)+(4,1)=(9,-2)$
• NEGATE 	• e.g. $-(-3,7)=(3,-7)$
• SUBTRACT 	• e.g. $(5,-3)-(4,1)=(1,-4)$
• SCALE 	• e.g. $-3(10,1)=(-30,-3)$.

Goal Extend these & other ideas to n-space ($\mathbb{R}^n$) using the encoding of vectors as n-tuples

$$(x_1,\dots,x_n) \quad \left(\text{or } [x_1 \ \cdots \ x_n] \text{ or } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)$$

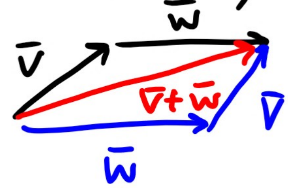
Basic Operations

If $\bar{v}=(v_1,\dots,v_n)$ & $\bar{w}=(w_1,\dots,w_n)$

then
- $\bar{v}+\bar{w}=(v_1+w_1, v_2+w_2,\dots,v_n+w_n)$.
- $-\bar{v}=(-v_1,\dots,-v_n)$
- $\bar{w}-\bar{v}=(w_1-v_1,\dots,w_n-v_n)$
- $k\bar{v}=(kv_1,\dots,kv_n)$, for $k\in\mathbb{R}$
- $\bar{0}=(0,\dots,0)$ called the <u>zero vector</u> $\left(=\underset{\text{0 scalar}}{0}\,\bar{u} \text{ for any } \bar{u}\right)$

Facts If $\bar{v}, \bar{w}, \bar{u} \in \mathbb{R}^n$, $k \in \mathbb{R}$, then

- $\bar{v} + \bar{w} = \bar{w} + \bar{v}$



- $(\bar{v} + \bar{w}) + \bar{u} = \bar{v} + (\bar{w} + \bar{u})$

So order of addition of vectors doesn't matter.

- $k(\bar{v} + \bar{w}) = k\bar{v} + k\bar{w}$
- $\bar{v} + (-\bar{v}) = \bar{0}$

For more see Textbook Theorem 3.1.1.

Definition $\bar{w} \in \mathbb{R}^n$ is a <u>linear combination</u> of $\bar{v}_1, \ldots, \bar{v}_r \in \mathbb{R}^n$ if there are scalars $k_1, \ldots, k_r \in \mathbb{R}$ with

$$\bar{w} = k_1\bar{v}_1 + k_2\bar{v}_2 + \ldots + k_r\bar{v}_r.$$

<u>coefficients</u>

Example Find scalars $k_1, k_2, k_3 \in \mathbb{R}$ such that $\bar{w} = (2, -1, 3)$

$$= k_1(3, 1, -1) + k_2(-1, 6, 2) + k_3(2, 1, 1)$$

Solution

$$= (3k_1, k_1, -k_1) + (-k_2, 6k_2, 2k_2) + (2k_3, k_3, k_3)$$

$$= \underbrace{(3k_1 - k_2 + 2k_3,\ k_1 + 6k_2 + k_3,\ -k_1 + 2k_2 + k_3)}_{\text{We could call this } \bar{u} \text{ \& apply } \downarrow}$$

Note $\bar{w} = \bar{u}$ exactly when $w_i = u_i$ for every $i = 1, \ldots, n$.

← for <u>any</u> vectors $\bar{w}$ & $\bar{u}$.

So need to solve:

$$\begin{aligned} 3k_1 - k_2 + 2k_3 &= 2 \\ k_1 + 6k_2 + k_3 &= -1 \\ -k_1 + 2k_2 + k_3 &= 3 \end{aligned}$$

Augmented matrix:

$$\begin{array}{c} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 2 \\ 1 & 6 & 1 & -1 \\ -1 & 2 & 1 & 3 \end{array}\right] \\ \begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{w} \end{array} \end{array}$$

Question Write $\bar{w}$

$= (2, -1, 3) = k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3$.

So we could go straight to the augmented matrix!

Row reduce:

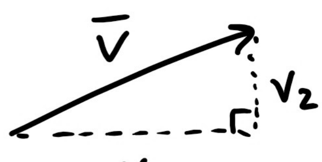
to get $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 13/5 \end{array}\right]$

So solution: $k_1 = -\frac{6}{5}$, $k_2 = -2/5$, $k_3 = 13/5$.

Check: $\bar{w} = k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3$.

3.2 Norm, Dot Product & Distance in $\mathbb{R}^n$

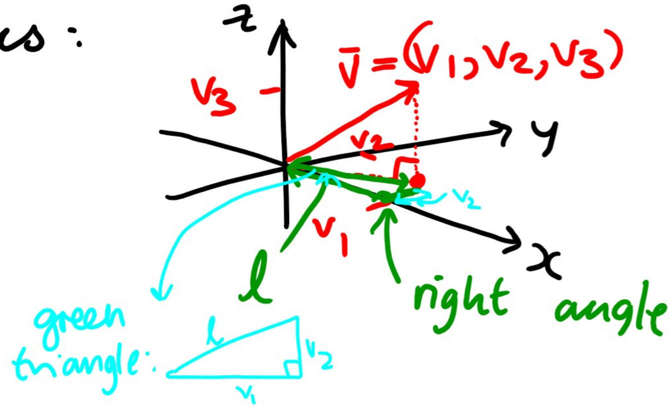
In $\mathbb{R}^2$



the magnitude of a vector $\bar{v} = (v_1, v_2)$ also called the length of $\bar{v}$ or the norm of $\bar{v}$

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$ Pythagoras also works:



$$\|\bar{v}\|^2 = l^2 + v_3^2$$
$$= (v_1^2 + v_2^2) + v_3^2$$
$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

__Definition__ In $\mathbb{R}^n$, the __norm__ (= __magnitude__ = __length__ = etc.) of a vector $\bar{v} = (v_1, v_2, \ldots, v_n)$ is

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

__Example__ $\|(-7, 5, 2, -1)\| = \sqrt{(-7)^2 + 5^2 + 2^2 + (-1)^2}$
$$= \sqrt{49 + 25 + 4 + 1}$$
$$= \sqrt{79}.$$

__Facts about Norms__ For any $\bar{v} \in \mathbb{R}^n$

(1) $\|\bar{v}\| \geqslant 0$ (2) $\|\bar{v}\| = 0$ iff $\bar{v} = \bar{0}$

(3) $\|k\bar{v}\| = |k|\,\|\bar{v}\|$, for any $k \in \mathbb{R}$.