1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 22

Yesterday NORMS IN $\mathbb{R}^n$

The <u>norm</u> (= magnitude = length) of a vector

$\bar{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is $\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$.

① $\|\bar{v}\| \geqslant 0$ always.

② $\|\bar{v}\| = 0$ if & only if $\bar{v} = \bar{0}$.

③ $\|k\bar{v}\| = |k|\,\|\bar{v}\|$.

<u>Unit vector</u> $\bar{v}$ with $\|\bar{v}\| = 1$

If $\bar{w} \neq \bar{0}$, can <u>normalize</u> $\bar{w}$: $\left(\frac{1}{\|\bar{w}\|}\right)\bar{w}$ $\quad$ lazy: $\left(= \frac{\bar{w}}{\|\bar{w}\|} \right)$

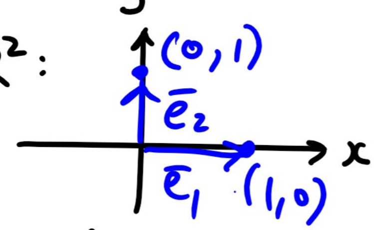
unit vector same direction as $\bar{w}$

<u>Special case</u> <u>Standard Unit Vectors in $\mathbb{R}^n$</u>

$$\left.\begin{aligned} \bar{e}_1 &= (1, 0, \ldots, 0) \\ \bar{e}_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ \bar{e}_n &= (0, \ldots, 0, 1) \end{aligned}\right\}$$

unit vectors in direction of coord. axes.

In $\mathbb{R}^2$:



Key Fact about standard unit vectors:

Any vector $\bar{v} = (v_1, \ldots, v_n)$ is a linear combination of $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$:

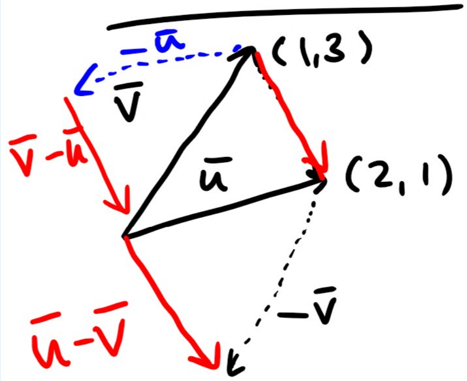
$$\bar{v} = v_1(1, 0, \ldots, 0) + \ldots + v_n(0, \ldots, 0, 1)$$
$$= v_1 \bar{e}_1 + v_2 \bar{e}_2 + \ldots + v_n \bar{e}_n.$$

Example

$$(3, 5, -27, 6) = 3\bar{e}_1 + 5\bar{e}_2 - 27\bar{e}_3 + 6\bar{e}_4$$
$$= 3(1,0,0,0) + 5(0,1,0,0) - 27(0,0,1,0) + 6(0,0,0,1)$$

Distance between Vectors

Example from $\mathbb{R}^2$: Find distance between $\bar{v} = (1,3)$ & $\bar{u} = (2,1)$



$$\text{Distance} = \|\bar{u} - \bar{v}\| = \|\bar{v} - \bar{u}\|$$
$$= \|(1,3) - (2,1)\| = \|(-1,2)\|$$
$$= \sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$

Distance between $\bar{u}$ & $\bar{v}$ in $\mathbb{R}^n$ is

$$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\| \qquad (= \|\bar{v} - \bar{u}\|)$$

(Makes sense only if $\bar{u}, \bar{v}$ in same $\mathbb{R}^n$!)

The dot product

In $\mathbb{R}^2$, $\mathbb{R}^3$ based on angle θ between vectors $\bar{u}, \bar{v}$:

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos(\theta)$$

In practice, compute $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2$ (in $\mathbb{R}^2$)

or $u_1 v_1 + u_2 v_2 + u_3 v_3$ (in $\mathbb{R}^3$)

Def^n For $\bar{u}, \bar{v}$ in $\mathbb{R}^n$, dot product of $\bar{u}$ & $\bar{v}$

is $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.

Example $(5, -1, 0, 3) \cdot (7, 0, -5, 2)$

$$= 5(7) + (-1)(0) + 0(-5) + 3(2)$$

$$= 35 + 6 = 41.$$

Matrix multiplication: lots of dot products!

If $\bar{u}, \bar{v}$ row matrices, then $\bar{u} \cdot \bar{v} = \bar{u}\bar{v}^T = \bar{v}\bar{u}^T$

$\bar{u} = [u_1 \, u_2 \cdots u_n]$, $\bar{v} = [v_1 \cdots v_n]$

Similar if $\bar{u}$ and/or $\bar{v}$ is column vector (p. 151).

Notice $\bar{v} \in \mathbb{R}^n$: $\bar{v} \cdot \bar{v} = v_1 v_1 + v_2 v_2 + \ldots + v_n v_n$

$$= v_1^2 + v_2^2 + \ldots + v_n^2$$

$$= \|\bar{v}\|^2$$

In fact $\|\bar{v}\| = \sqrt{\bar{v}\cdot\bar{v}}$

Example Suppose $\|\bar{u}\| = 2$, $\|\bar{v}\| = 3$, $\bar{u}\cdot\bar{v} = -6$.
Find $d(\bar{u},\bar{v})$.

Solution $= \|\bar{u}-\bar{v}\| = \sqrt{(\bar{u}-\bar{v})\cdot(\bar{u}-\bar{v})}$

$= \sqrt{\bar{u}\cdot\bar{u} - \bar{v}\cdot\bar{u} - \bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v}}$

$= \sqrt{\|\bar{u}\|^2 - 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2}$

$= \sqrt{2^2 - 2(-6) + 3^2}$

$= \sqrt{25} = 5$.

Dot products in $\mathbb{R}^n$ behave just like regular products of real #s.

Recall $\bar{u}\cdot\bar{v} = \|\bar{u}\|\|\bar{v}\|\cos\theta$ in $\mathbb{R}^2, \mathbb{R}^3$

$\uparrow$ θ angle between $\bar{u}$ & $\bar{v}$.

Define angle between $\bar{u}$ & $\bar{v}$ in $\mathbb{R}^n$ to be

$$\theta = \cos^{-1}\left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}\right)$$

Careful: is this in $[-1, 1]$? Yes:

Fact: Cauchy-Schwarz Inequality

$$|\bar{u}\cdot\bar{v}| \leq \|\bar{u}\|\|\bar{v}\|$$ for $\bar{u},\bar{v}$ in $\mathbb{R}^n$

Example Find the angle between $\bar{u} = (1, 1, 0, 2)$ & $\bar{v} = (-1, 0, 0, -1)$

Solution

$$\bar{u} \cdot \bar{v} = -1 + 0 + 0 - 2 = -3$$

$$\|\bar{u}\| = \sqrt{1+1+4} = \sqrt{6} = \sqrt{2}\sqrt{3}$$

$$\|\bar{v}\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

So
$$\cos(\theta) = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \frac{-3}{\sqrt{2}\sqrt{3}\sqrt{2}} = -\frac{\sqrt{3}}{2}$$

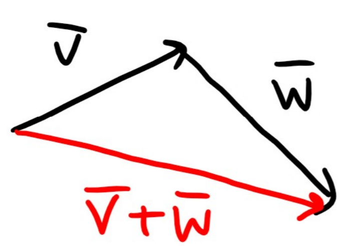
$$\Rightarrow \theta = \frac{5\pi}{6}.$$

For all $\bar{v}, \bar{w} \in \mathbb{R}^n$:

① **Triangle Inequality**

$$\|\bar{v} + \bar{w}\| \leq \|\bar{v}\| + \|\bar{w}\|$$

In $\mathbb{R}^2$



② **Parallelogram Equation**

$$\|\bar{v} + \bar{w}\|^2 + \|\bar{w} - \bar{v}\|^2 = 2\left(\|\bar{v}\|^2 + \|\bar{w}\|^2\right)$$

$$(\bar{v}+\bar{w})\cdot(\bar{v}+\bar{w}) + (\bar{w}-\bar{v})\cdot(\bar{w}-\bar{v})$$

$$= \|\bar{v}\|^2 + \cancel{2\bar{v}\cdot\bar{w}} + \|\bar{w}\|^2 + \|\bar{w}\|^2 - \cancel{2\bar{v}\cdot\bar{w}} + \|\bar{v}\|^2$$

$$= 2(\|\bar{v}\|^2 + \|\bar{w}\|^2)$$

(3) $\bar{v} \cdot \bar{w} = \frac{1}{4}\|\bar{v}+\bar{w}\|^2 - \frac{1}{4}\|\bar{w}-\bar{v}\|^2$

↳ Try it as an Exercise
(see p. 150; 3.2.7).