# 1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 23

## Last Time

### Dot Product & Angle

For $\bar{u}, \bar{v}$ in $\mathbb{R}^n$, $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos(\theta)$

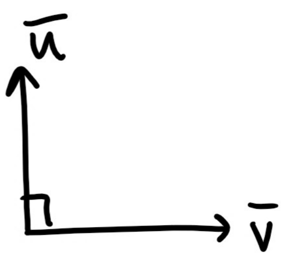
this formula now defines the angle between $\bar{u}$ & $\bar{v}$

while $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$ defines the dot product

Facts * $\|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}}$ ; * $\cdot$ behaves like multiplication of reals;
* $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$ ; * $|\bar{u} \cdot \bar{v}| \leq \|\bar{u}\| \|\bar{v}\|$ ; * $\|\bar{u} + \bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$.

## 3.3 Orthogonality – generalizes "perpendicular"

$\bar{u}, \bar{v}$




Angle $90° = \frac{\pi}{2}$

& $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos\left(\frac{\pi}{2}\right)$

$= 0$

**Definition** $\bar{u}, \bar{v} \in \mathbb{R}^n$ are orthogonal iff $\bar{u} \cdot \bar{v} = 0$

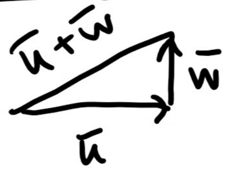
**Example** $\bar{u} = (1, -1, 3, 2)$ , $\bar{v} = (7, -5, -2, -3)$

$\bar{u} \cdot \bar{v} = 7 + 5 - 6 - 6 = 0.$

**Important Example** Standard unit vectors $\bar{e}_1, \ldots, \bar{e}_n$

– all orthogonal i.e. $\bar{e}_i \cdot \bar{e}_j = 0$, $i \neq j$

## Pythagoras



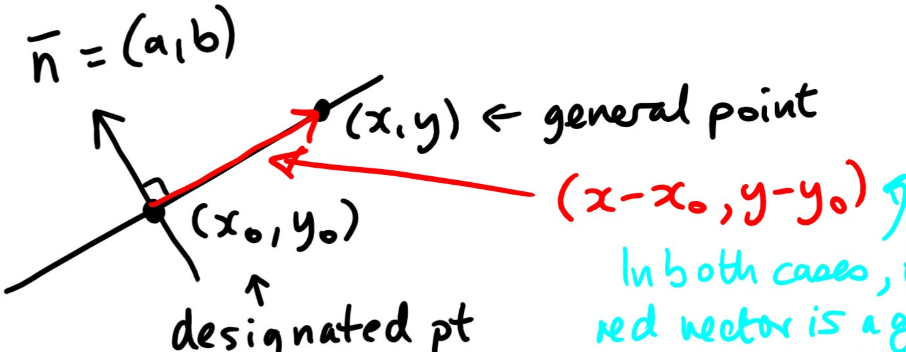

$\|\bar{u}+\bar{w}\|^2 = \|\bar{u}\|^2 + \|\bar{w}\|^2$, <u>if</u> $\bar{u},\bar{w}$ orthogonal.

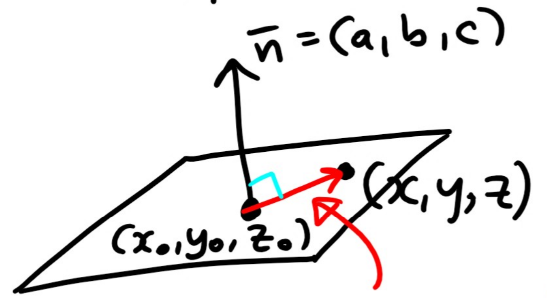
$\langle \bar{u}+\bar{w}, \bar{u}+\bar{w}\rangle = \|\bar{u}\|^2 + \underbrace{2\langle \bar{u},\bar{w}\rangle}_{=0 \text{ as } \bar{u},\bar{w} \text{ orthogonal!}} + \|\bar{w}\|^2$

(Exercise)

## Lines in $\mathbb{R}^2$ & Planes in $\mathbb{R}^3$

→ described uniquely by a point & a vector called the <u>normal</u> $\bar{n}$ orthogonal to the line/plane




In both cases, the red vector is a general vector lying in the line/plane

we have $\bar{n}\cdot(x-x_0, y-y_0) = 0$

$(a,b)\cdot(x-x_0, y-y_0) = 0$

$ax + by + (-ax_0 - by_0) = 0$

$(x-x_0, y-y_0, z-z_0)$

$0 = (a,b,c)\cdot(x-x_0, y-y_0, z-z_0)$

$= ax + by + cz + (-ax_0 - by_0 - cz_0)$

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So $ax+by+c=0$ is a line in $\mathbb{R}^2$ with normal $\bar{n}=(a,b)$ & $ax+by+cz+d=0$ is a plane in $\mathbb{R}^3$ with normal $\bar{n}=(a,b,c)$.

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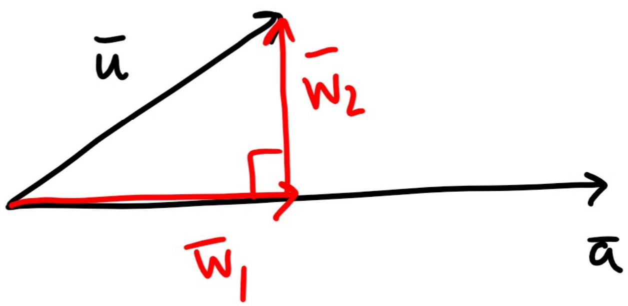
<u>Projection Theorem</u> If $\bar{u}, \bar{a} \in \mathbb{R}^n$, $\bar{a} \neq \bar{0}$, then




$\bar{u} = \bar{w}_1 + \bar{w}_2$ where $\bar{w}_1 = k\bar{a}$ (some k TBD)

& $\bar{w}_2 \cdot \bar{a} = 0$.

To find $\bar{w}_1$ & $\bar{w}_2$:

$$\bar{u} = \bar{w}_1 + \bar{w}_2 = k\bar{a} + \bar{w}_2$$

So $\bar{u} \cdot \bar{a} = k\,\bar{a}\cdot\bar{a} + \cancelto{0}{\bar{w}_2 \cdot a}$

$$\Rightarrow k = \frac{\bar{u}\cdot\bar{a}}{\|\bar{a}\|^2}$$

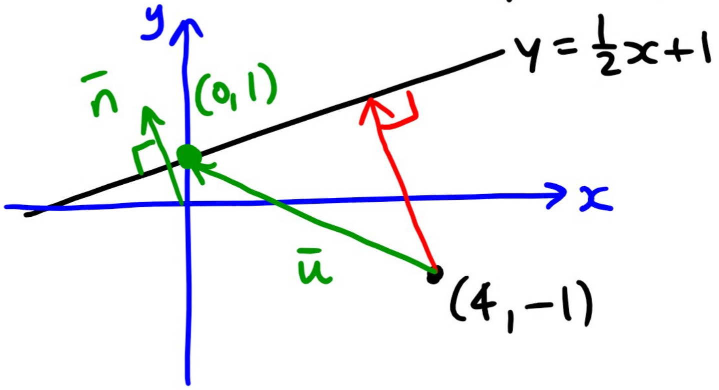
So $\bar{w}_1 = \underbrace{\overbrace{\left(\frac{\bar{u}\cdot\bar{a}}{\|\bar{a}\|^2}\right)}^{k}\bar{a}}$

called the <u>orthogonal projection</u> of $\bar{u}$ onto $\bar{a}$: $\text{proj}_{\bar{a}}\bar{u}$

& $\bar{w}_2 = \bar{u} - \bar{w}_1 = \underbrace{\bar{u} - \left(\frac{\bar{u}\cdot\bar{a}}{\|\bar{a}\|^2}\right)\bar{a}}$

component of $\bar{u}$ orthogonal to $\bar{a}$

<u>Example</u> Find the shortest distance from ~~(-4, 1)~~ (4, -1) to the line $y = \frac{1}{2}x + 1$.




red vector

Answer $= \left\| \text{proj}_{\bar{n}}\bar{u} \right\|$

$$= \left\| \frac{\bar{u}\cdot\bar{n}}{\|\bar{n}\|^2}\,\bar{n} \right\|$$

$y = \frac{1}{2}x + 1$

$\rightarrow x - 2y + 2 = 0$

$\bar{n} = (1, -2)$

$\bar{u} = (0,1) - (4,-1) = (-4,2)$

We pick any point on the line that we know e.g. (0,1) to place $\bar{u}$

$$= \left| \frac{\bar{u}\cdot\bar{n}}{\|\bar{n}\|^{\cancel{2}}} \right| \cancel{\|\bar{n}\|} = \frac{|\bar{u}\cdot\bar{n}|}{\|n\|}$$

$$= \frac{|(-4,2)\cdot(1,-2)|}{\sqrt{1^2 + (-2)^2}}$$

$$= \frac{|-8|}{\sqrt{5}} = \frac{8}{\sqrt{5}}. \quad //$$

**Shortcut:** in $\mathbb{R}^2$ the shortest distance from $(x_0, y_0)$ to $ax + by + c = 0$

is $$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

$\bar{n} = (a,b)$

$\bar{u} = (0, -\frac{c}{b}) - (x_0, y_0)$

$= (-x_0, -\frac{c}{b} - y_0)$

Point we know: where line crosses y-axis

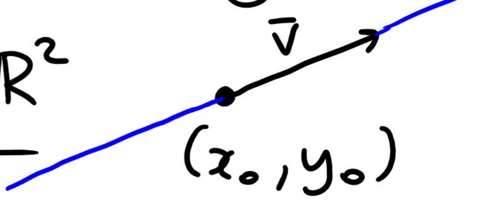
In $\mathbb{R}^3$, the shortest distance from $(x_0, y_0, z_0)$ to $ax + by + cz + d = 0$

is $$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

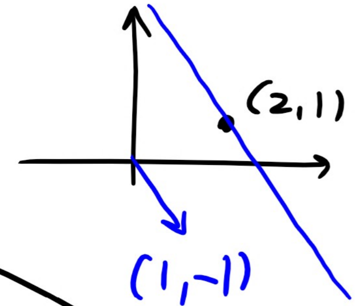
## 3.4 The Geometry of Linear Systems

↳ another way to think about lines/planes

**Lines in $\mathbb{R}^2$**




Every point on line is $(x_0, y_0) + t\bar{v}$ for some $t \in \mathbb{R}$




<u>Example</u> $(2,1) + t(1,-1)$

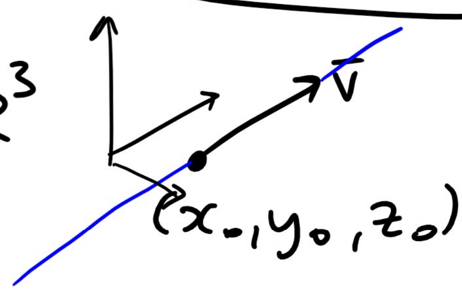
Points on line $(x,y) = (2+t,\ 1-t)$

"<u>parametric form</u>" of line

<u>Notice</u>: $x + y = 3$ so this line is the

Solution set to this LE

<u>Lines in $\mathbb{R}^3$</u>




Parametrically

$$(x,y,z) = (x_0, y_0, z_0) + t\bar{v},\quad t \in \mathbb{R}$$

<u>Example</u> Point $(1,-1,3)$

& vector $(-2,1,1)$ trace out line

with points $(x,y,z) = (1-2t,\ -1+t,\ 3+t)$

This is the solution set to $\begin{cases} x + 2y = -1 \\ x + 2z = 7. \end{cases}$