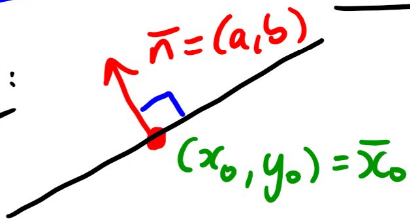
1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 24

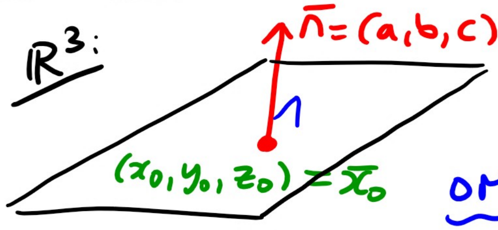
Last Time

Lines & Planes using vectors

$\mathbb{R}^2$:



$\mathbb{R}^3$:



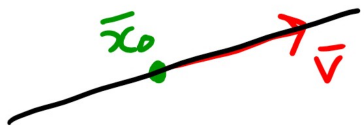
→ With a specified point and an orthogonal normal vector

$\bar{u}, \bar{v} \in \mathbb{R}^n$ orthogonal: $\bar{u} \cdot \bar{v} = 0$

Line Equation:
$$ax + by + c = 0$$

Plane Equation:
$$ax + by + cz + d = 0$$

In $\mathbb{R}^2$ & $\mathbb{R}^3$:



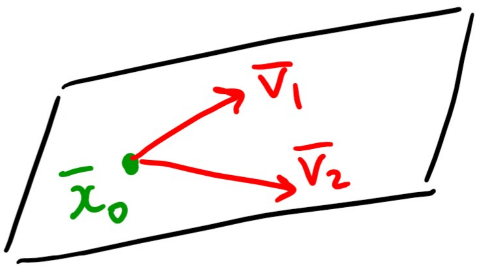
(x, y) or (x, y, z)

(x_0, y_0) or (x_0, y_0, z_0)

Parametric Line Equation: $\bar{x} = \bar{x}_0 + t\bar{v}, \; t \in \mathbb{R}.$

(v_1, v_2) or (v_1, v_2, v_3)

In $\mathbb{R}^3$:



$\bar{v}_1$ & $\bar{v}_2$ NOT colinear (i.e. one is not a scalar multiple of the other)

Points on plane

$$\bar{x} = \bar{x}_0 + t\bar{v}_1 + s\bar{v}_2 \quad , t, s \in \mathbb{R}$$

In $\mathbb{R}^n$:

The line through $\bar{x}_0$ parallel to vector $\bar{v}$ is $\bar{x} = \bar{x}_0 + t\bar{v}$, $t \in \mathbb{R}$

The plane through $\bar{x}_0$ parallel to $\bar{v}_1, \bar{v}_2$ is $\bar{x} = \bar{x}_0 + t\bar{v}_1 + s\bar{v}_2$, $t, s \in \mathbb{R}$.

$\bar{v}_1$ & $\bar{v}_2$ NOT COLINEAR

Line Equation in $\mathbb{R}^n$ is solution set to system of n-1 linear equations ~~variables~~

Plane " " " " " " " " n-2 linear equations ~~variables~~

n variables!

<u>Example</u> Find the line in $\mathbb{R}^5$ passing through $(1, 0, -1, 2, 3)$ and $(5, -1, -2, 0, 6)$.

<u>Solution</u> Need a point on line $\bar{x}_0$ & a vector direction $\bar{v}$

Pick one, say $\bar{x}_0 = (1, 0, -1, 2, 3)$ ←

Get $\bar{v} = (1, 0, -1, 2, 3) - (5, -1, -2, 0, 6)$

$= (-4, 1, 1, 2, -3)$. ←

So line: $\bar{x} = \bar{x}_0 + t\bar{v}$, $t \in \mathbb{R}$

$(x_1, x_2, x_3, x_4, x_5) = (\underbrace{1-4t}_{x_1=}, \underbrace{t}_{x_2=}, \underbrace{-1+t}_{x_3=}, \underbrace{2+2t}_{x_4=}, \underbrace{3-3t}_{x_5=})$

<u>The geometry of linear systems</u>

Say $A\bar{x} = \bar{0}$

A is $\underline{\underline{m}} \times n$

m equations like: $\underbrace{a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n}_{\bar{a}_i \cdot \bar{x}} = 0$

where $\bar{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ is ith row of A.

i.e. every solution $\bar{x}$ to $A\bar{x} = \bar{0}$ is orthogonal to every row of A

<u>Example</u> If A is 2×3 & $A\bar{x} = \bar{0}$

2 equations $= 3 - 1$; 3 variables

Get a line (by above) for solution set in $\mathbb{R}^3$

For sure $\bar{x} = \bar{0}$ is a solution so line goes through the origin.

So solution set looks like $t\bar{v}$, $t \in \mathbb{R}$ where $\bar{v}$ is orthogonal to the rows of A.

Now suppose $A\bar{x} = \bar{b}$ & we find <u>one</u> solution $\bar{x}_0$ i.e. $A\bar{x}_0 = \bar{b}$.

Then all solutions look like $\bar{x} = \bar{x}_0 + \bar{w}$ * Why? See end of file.
where $\bar{w}$ is a solution to $A\bar{x} = \bar{0}$ i.e. $A\bar{w} = \bar{0}$.

In earlier example, A 2×3, now $A\bar{x} = \bar{b}$

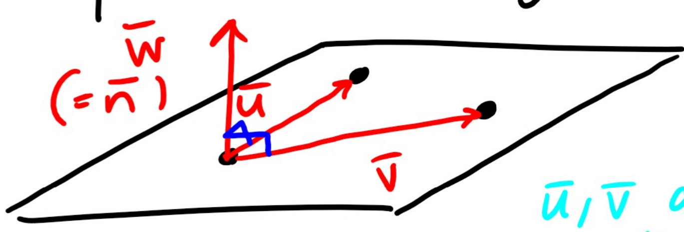
Solutions to $A\bar{x} = \bar{0}$ are $t\bar{v}$, $t \in \mathbb{R}$ so if we have one solution $\bar{x}_0$ to $A\bar{x}=\bar{b}$, then all solutions to $A\bar{x}=\bar{b}$ look like $\bar{x}_0+t\bar{v}$, $t\in\mathbb{R}$.

Notice, this is a line parallel to the solution set to $A\bar{x}=\bar{0}$, which was $t\bar{v}$, $t\in\mathbb{R}$.

3.5 Cross Product — <u>only</u> in $\mathbb{R}^3$

$\bar{u},\bar{v}\in\mathbb{R}^3$, want $\bar{w}$ orthogonal to both $\bar{u}$ & $\bar{v}$

- useful for finding equation of a plane with e.g. 3 points given



Notice: don't need $\bar{u},\bar{v}$ orthogonal, since as long as $\bar{u},\bar{v}$ are in the plane & $\bar{w}$ is orthogonal to both, it will be orthogonal i.e. normal to the plane.

The <u>cross product</u> of $\bar{u}=(u_1,u_2,u_3)$ & $\bar{v}=(v_1,v_2,v_3)$ is

$$\bar{u}\times\bar{v}=\begin{vmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\begin{matrix}\\ \leftarrow\bar{u}\\ \leftarrow\bar{v}\end{matrix}$$

Sometimes $\bar{e}_1,\bar{e}_2,\bar{e}_3$ called $\bar{i},\bar{j},\bar{k}$.

<u>Check:</u> $\bar{e}_1\times\bar{e}_2=\bar{e}_3$
$\bar{e}_2\times\bar{e}_3=\bar{e}_1$
$\bar{e}_3\times\bar{e}_1=\bar{e}_2$.

Not a matrix! So not actually a determinant

$$=\bar{e}_1\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}-\bar{e}_2\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}+\bar{e}_3\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}$$

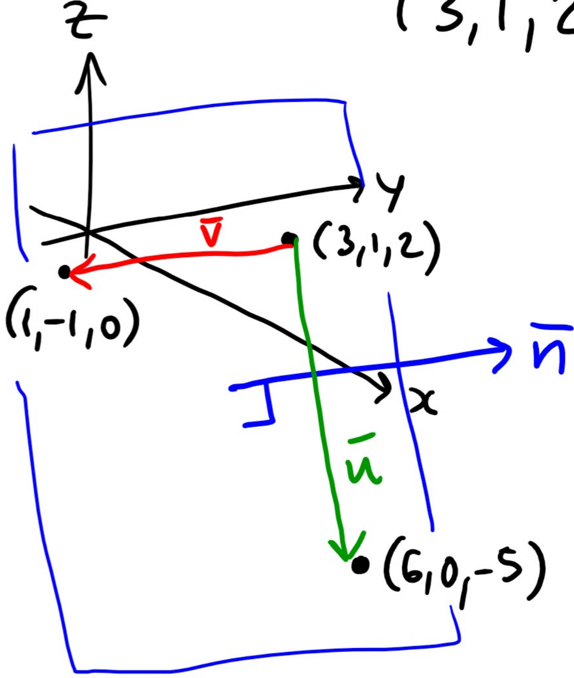
Pretending nevertheless that the above expression is a determinant.

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \; - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \; \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= \left(u_2 v_3 - u_3 v_2, \; -(u_1 v_3 - u_3 v_1), \; u_1 v_2 - u_2 v_1 \right)$$

$$\bar{u} \cdot (\bar{u} \times \bar{v}) = 0 = \bar{v} \cdot (\bar{u} \times \bar{v}) \qquad \text{(Check!)}$$

<u>Example</u> Find the plane containing $(3,1,2)$, $(6,0,-5)$, $(1,-1,0)$.



$$\bar{u} = (6,0,-5) - (3,1,2) = (3,-1,-7)$$

$$\bar{v} = (1,-1,0) - (3,1,2) = (-2,-2,-2)$$

$$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ 3 & -1 & -7 \\ -2 & -2 & -2 \end{vmatrix}$$

$$= -12\bar{e}_1 + 20\bar{e}_2 - 8\bar{e}_3$$

$$= (-12, 20, -8)$$

Plane equation: $-12x + 20y - 8z + d = 0$

To find d, plug in one of the points e.g.

$(3,1,2)$: $-36+20-16+d=0$

$$\Rightarrow d = 32.$$

Equation: $-12x + 20y - 8z + 32 = 0.$

Properties of cross products

$$\begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

- $\bar{u} \times \bar{u} = \bar{0}$
- $\bar{u} \times \bar{0} = \bar{0}$
- $\bar{u} \times \bar{v} = -(\bar{v} \times \bar{u})$ (right-hand rule)
- $\bar{u} \times (\bar{v} + \bar{w}) = (\bar{u} \times \bar{v}) + (\bar{u} \times \bar{w})$
- $(k\bar{u}) \times \bar{v} = k(\bar{u} \times \bar{v})$

- (Lagrange's Identity) – multiply out to get

$$\|\bar{u} \times \bar{v}\| = \sqrt{\|\bar{u}\|^2 \|\bar{v}\|^2 - (\bar{u} \cdot \bar{v})^2}$$

(Check!)

See Theorems 3.5.1 & 3.5.2 in textbook for more.

Here is an explanation for the claim

"All solutions $\bar{x}$ to $A\bar{x}=\bar{b}$ look like $\bar{x}=\bar{x}_0+\bar{w}$, where $\bar{x}_0$ is a known solution (i.e. $A\bar{x}_0=\bar{b}$) & $\bar{w}$ is some solution to the homogeneous system i.e. $A\bar{w}=\bar{0}$."

First take a vector that looks like $\bar{x}=\bar{x}_0+\bar{w}$, as described above. Then $A\bar{x} = A(\bar{x}_0+\bar{w}) = A\bar{x}_0 + A\bar{w} = \bar{b}+\bar{0} = \bar{b}$, so $\bar{x}=\bar{x}_0+\bar{w}$ <u>is</u> in fact a solution to $A\bar{x}=\bar{b}$.

Now consider $\bar{x}$ & write it as $\bar{x} = \bar{x}_0 + (\bar{x}-\bar{x}_0)$.

If $A\bar{x}=\bar{b}$, then $A(\bar{x}-\bar{x}_0) = A\bar{x}-A\bar{x}_0 = \bar{b}-\bar{b} = \bar{0}$, so $\bar{x}-\bar{x}_0$ is a solution to the homogeneous system i.e. setting $\bar{w}=\bar{x}-\bar{x}_0$ we see that $\bar{x}$ has the required form, $\bar{x}=\bar{x}_0+\bar{w}$, with $A\bar{w}=\bar{0}$.