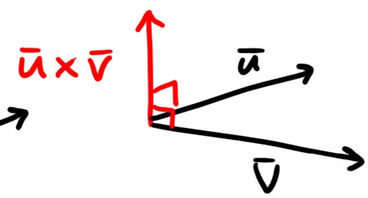
1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19, C03) Lecture 25

Yesterday: Cross Product in $\mathbb{R}^3$

$$\bar{u} \times \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(u_2v_3 - u_3v_2,\ -(u_1v_3 - u_3v_1),\ u_1v_2 - u_2v_1\right)$$

$$(\bar{u} \times \bar{v}) \cdot \bar{u} = 0 = (\bar{u} \times \bar{v}) \cdot \bar{v}$$

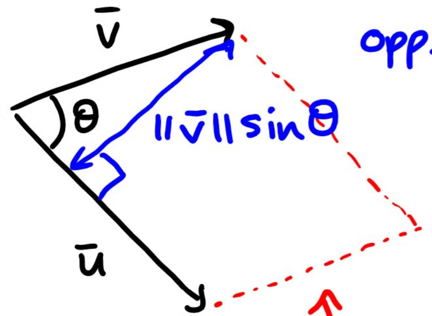


- $\bar{u} \times \bar{v} = -(\bar{v} \times \bar{u})$; • $\bar{u} \times \bar{0} = \bar{0}$; • $\bar{u} \times \bar{u} = \bar{0}$; etc. (see 3.5.1, 3.5.2)

- Lagrange's Identity: $\|\bar{u} \times \bar{v}\| = \sqrt{\|\bar{u}\|^2 \|\bar{v}\|^2 - (\bar{u} \cdot \bar{v})^2}$

$$(\bar{u} \cdot \bar{v})^2 = \|\bar{u}\|^2 \|\bar{v}\|^2 \cos^2\theta$$

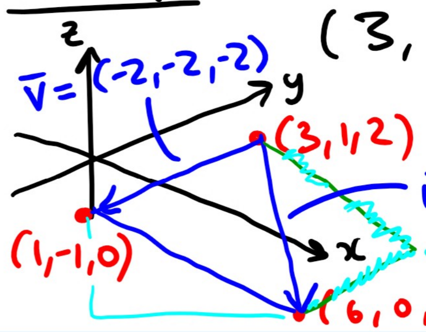
$$\|\bar{u} \times \bar{v}\| = \sqrt{\|\bar{u}\|^2 \|\bar{v}\|^2 - \|\bar{u}\|^2 \|\bar{v}\|^2 \cos^2\theta}$$
$$= \|\bar{u}\| \|\bar{v}\| \sqrt{1 - \cos^2\theta}$$
$$= \|\bar{u}\| \|\bar{v}\| \sin\theta$$



opp. = hyp. $\times \sin\theta$

Area of parallelogram
= base × height
= $\|\bar{u}\| \times \|\bar{v}\| \sin\theta$
= $\|\bar{u} \times \bar{v}\|$.

Examples Find the area of the triangle joining $(3, 1, 2)$, $(6, 0, -5)$, $(1, -1, 0)$



← If you want to use this parallelogram, you should use (1,-1,0) → (3,1,2), (1,-1,0) → (6,0,-5) — gives same answer, of course.

Area of $\Delta = \frac{1}{2}$ (area of parallelogram given by $\bar{u}$ & $\bar{v}$)

$$= \frac{1}{2}\ \|\bar{u}\times\bar{v}\| = \frac{1}{2}\|(-12,20,-8)\|$$

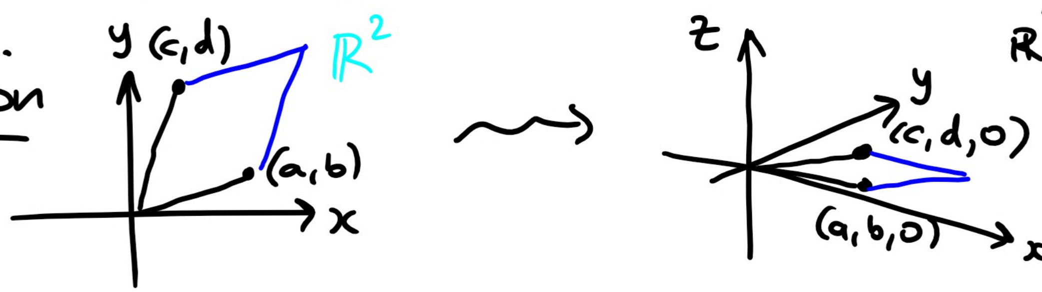
see last lecture

$$= \frac{1}{2}\sqrt{(-12)^2+20^2+(-8)^2} = \frac{1}{2}\sqrt{608}$$

$$= \frac{1}{2}\left(4\sqrt{38}\right) \cong \underline{\underline{12.33}}$$

Example Find the area of the parallelogram in $\mathbb{R}^2$ given by $(a,b),\ (c,d) \in \mathbb{R}^2$.

Solution



$$\text{Area} = \|(a,b,0)\times(c,d,0)\|$$

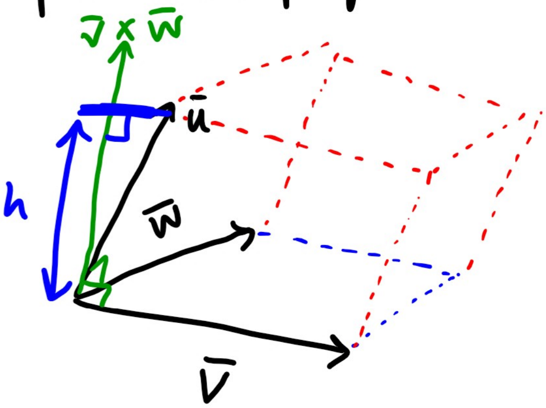
$$= \left\| \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ a & b & 0 \\ c & d & 0 \end{vmatrix} \right\| = \|(0,0,ad-bc)\|$$

$$= |ad-bc|$$

$$= \left| \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|.$$

Example Find the volume of the parallelipiped in $\mathbb{R}^3$ given by $\bar{u},\bar{v},\bar{w}$.

Solution



Volume = area of base × height

$$= \|\bar{v}\times\bar{w}\| \times$$

$$\|\text{proj}_{\bar{v}\times\bar{w}}\,\bar{u}\|$$

$$\text{height} = \|\operatorname{proj}_{\bar{v}\times\bar{w}} \bar{u}\| = \frac{|\bar{u}\cdot(\bar{v}\times\bar{w})|}{\|\bar{v}\times\bar{w}\|}$$

$$\downarrow \quad \frac{\cancel{\|\bar{v}\times\bar{w}\|}\,|\bar{u}\cdot(\bar{v}\times\bar{w})|}{\cancel{\|\bar{v}\times\bar{w}\|}}$$

$$= |\underbrace{\bar{u}\cdot(\bar{v}\times\bar{w})}|$$

The <u>scalar triple product</u> of $\bar{u}$, $\bar{v}$ & $\bar{w}$.

<u>Notice</u> $\bar{u}\cdot(\bar{v}\times\bar{w}) = \bar{u}\cdot\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right)$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

4.1 Real Vector Spaces

- generalizes "the collection of all vectors in $\mathbb{R}^n$" & their interactions

favourite n

- Q: What other collections of mathematical "objects" interact in the same ways

↳ can add them, subtract, negate & can scale them by a real #

↳ cannot multiply them (& get another vector)

↳ also interesting phenomena: dot product, norm, distance, ...

<u>Definition</u> V – real vector space

– non-empty collection of (any kind of !!!) math. objects with concepts of

→ addition of objects

→ scalar multiplication of objects

↳ by $k \in \mathbb{R}$

Satisfying 10 <u>axioms</u> (rules):

Notice how, in the absence of knowing which objects we're talking about as members of V & thus knowing what some more appropriate notation might be), our default notation is $\bar{u}, \bar{v}$ etc. & we call the objects "vectors", BUT this does NOT mean the objects are necessarily line segments!

1. If $\bar{u}, \bar{v} \in V$, then so is $\bar{u} + \bar{v} \in V$ ("V is <u>closed under addition</u>")

2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$

3. $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$

} order of addition doesn't matter

4. There is an object called $\bar{0}$, the "<u>zero vector</u>" with $\bar{u} + \bar{0} = \bar{u} = \bar{0} + \bar{u}$

5. For each $\bar{u} \in V$, there's a "<u>negative of $\bar{u}$</u>", $-\bar{u}$ with $\bar{u} + (-\bar{u}) = \bar{0}$

6. If $k \in \mathbb{R}$, $\bar{v} \in V$, then $k\bar{v} \in V$
("V is <u>closed under scalar multiplication</u>")

7. $k(\bar{u} + \bar{v}) = k\bar{u} + k\bar{v}$

8. $(k+m)\bar{u} = k\bar{u} + m\bar{u}$
addition of real #s

9. $(km)\bar{u} = k(m\bar{u})$
mult. of real #s

10. $1\bar{u} = \bar{u}$

<u>Examples</u> ① $\mathbb{R}^n$ for any fixed n
with usual addition & scalar multiplication of vectors.

② $V = M_{m\times n}(\mathbb{R})$, the set of all $m\times n$ matrices with matrix addition & matrix scalar mult.

fixed choice of m, n
if $m=n$, write $M_n(\mathbb{R})$

To check: need to check all 10 axioms

e.g. $m\times n$ matrices A, B : 1. Is $A+B$ an $m\times n$ matrix? ✓

2. Is $A+B = B+A$? ✓ if both A & B $m\times n$

Let's finish this off:

3. Is $A + (B+C) = (A+B)+C$ if A, B, C all $m \times n$? ✓

4. What is the "zero vector" here? The $m \times n$ matrix which, when you add it to any matrix A, gives A back?
The $m \times n$ zero matrix $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, of course. ✓

5. If A is an $m \times n$ matrix, what is $-A$? i.e. which matrix satisfies $A + (-A) = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$?

Well $-A = (-1)A$, the matrix scalar multiple of A by -1, would do the job.
$$0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix};$$ ✓

6. If $k \in \mathbb{R}$ and A is an $m \times n$ matrix, is kA also an $m \times n$ matrix? Yes ✓

7. Is $k(A+B) = kA + kB$, if A, B are $m \times n$ matrices & $k \in \mathbb{R}$? Yes ✓

8. Is $(k+m)A = kA + mA$ if $k, m \in \mathbb{R}$ & A is an $m \times n$ matrix? Yes ✓

9. Is $(km)A = k(mA)$ if $k, m \in \mathbb{R}$ & A is an $m \times n$ matrix? Yes ✓

10. Is $(1)A = A$ if A is an $m \times n$ matrix? Yes ✓

(Check rules about working with matrices to check 2., 3., 7., 8., 9., 10.)