1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 26

Last Time

Real Vector Spaces V

→ non-empty collection of objects (written as $\bar{u}, \bar{v}$ etc.)

with $+$ → addition of objects

$k.$ → scalar multiplication of objects by real numbers k

satisfying AXIOMS:

A1) For all $\bar{u}, \bar{v} \in V$, $\bar{u}+\bar{v} \in V$. ← "Closure under addition"

A2) For all $\bar{u}, \bar{v} \in V$, $\bar{u}+\bar{v} = \bar{v}+\bar{u}$.

A3) For all $\bar{u}, \bar{v}, \bar{w} \in V$, $\bar{u}+(\bar{v}+\bar{w}) = (\bar{u}+\bar{v})+\bar{w}$.

A4) There is a zero $\bar{0}$ with $\bar{u}+\bar{0} = \bar{0}+\bar{u} = \bar{u}$, for all $\bar{u} \in V$.

A5) For all $\bar{u} \in V$, there is a negative $-\bar{u}$ with $\bar{u} + (-\bar{u}) = \bar{0}$.

A6) For all $\bar{u} \in V$, $k \in \mathbb{R} \Rightarrow k.\bar{u} \in V$. "closure under scalar mult."

A7) For all $k \in \mathbb{R}$, $\bar{u}, \bar{v} \in V$, $k.(\bar{u}+\bar{v}) = k.\bar{u} + k.\bar{v}$

A8) For all $k, m \in \mathbb{R}$, $\bar{u} \in V$, $(k+m).\bar{u} = k.\bar{u} + m.\bar{u}$.

A9) For all $k, m \in \mathbb{R}$, $\bar{u} \in V$, $(k.m).\bar{u} = k.(m.\bar{u})$.

A10) For all $\bar{u} \in V$, $1.\bar{u} = \bar{u}$.

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③ $V = \{\text{one single objects}\} = \{\bar{0}\}$

Define
$$\bar{0} + \bar{0} = \bar{0}$$
$$k\bar{0} = \bar{0} \quad , \text{ for any } k \in \mathbb{R}$$

Check e.g. (A9) $(km)\bar{0} = \bar{0} = k(\bar{0}) = k(m\bar{0})$ ✓

④ $F(-\infty, \infty) = \{$ all $\underbrace{\text{real-valued}}_{\text{range is in } \mathbb{R}}$ functions with domain $(-\infty, \infty)\}$

"$F(\mathbb{R})$"

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

To define $+$: $[f+g] : \mathbb{R} \longrightarrow \mathbb{R}$; $[f+g](x) = f(x) + g(x)$

& scalar mult. : $[kf] : \mathbb{R} \longrightarrow \mathbb{R}$; $[kf](x) = kf(x)$

Check e.g. (A7) $k(\bar{u}+\bar{v}) = k\bar{u} + k\bar{v}$

here : $k[f+g] = kf + kg$

Have to check: $\big[k[f+g]\big](x) = [kf+kg](x)$ for every $x \in \mathbb{R}$.

$$\big(k[f+g]\big)(x) \underset{\substack{\uparrow \\ \text{def. of} \\ \text{scalar mult.}}}{=} k[f+g](x) \underset{\substack{\uparrow \\ \text{def. of} \\ \text{addition}}}{=} k\big(f(x)+g(x)\big) \underset{\substack{\uparrow \\ \text{prop.} \\ \text{of real} \\ \text{numbers}}}{=} k\big(f(x)\big) + k\big(g(x)\big)$$

$$\underset{\substack{\nearrow \\ \text{def. of} \\ \text{scalar mult.}}}{=} [kf](x) + [kg](x) \underset{\substack{\uparrow \\ \text{def. of addition}}}{=} [kf+kg](x).$$

(5) P_d , the set of all polynomials of degree $\leq d$ defined on $\mathbb{R}$ $[a_0 + a_1 x + \ldots + a_d x^d, a_i \in \mathbb{R}]$

– same addition & scalar mult. as in (4) $F(-\infty, \infty)$.

(6) $\mathbb{R}$: here define $u + v = uv$ for $u, v \in \mathbb{R}$ ← usual multiplication of real #s

e.g. $3 + 2 = 6$

& $ku = u^k$ ← usual exponent for $u \in \mathbb{R}$ (space), $k \in \mathbb{R}$ (scalar)

↑
see textbook p. 188 Ex. 8

$$\text{e.g. } 2(5) = 5^2 = 25$$

scale 5 by scalar 2

So sometimes usual objects are vector spaces with wacky operations.

Or not:

<u>Non-Examples</u> (A) Textbook p. 188 Ex. 7

$V = \mathbb{R}^2 \qquad \bar{u} + \bar{v}$ – usual addition

$$k\bar{u} = (ku_1, 0)$$

(A1)–(A9) all OK.

(A10): $1\bar{u} = \bar{u}$: to illustrate failure, need an explicit example;

any $\bar{u} = (u_1, u_2)$ with $u_2 \neq 0$ will do:

$$\text{e.g. } 1(2,5) = (2, 0) \neq (2,5) \ \times.$$

(B) $V = \mathbb{R}^2$ usual scalar mult.

$$\bar{u} + \bar{v} = (\underline{|u_1| + |v_1|}, \underline{|u_2| + |v_2|})$$

always non-negative

e.g. A4 fails: if $\bar{0} = (v_1, v_2)$

then, say, $(-2,3) + \bar{0}$

$$= (\underbrace{2 + |v_1|}_{\geq 2}, \underbrace{3 + |v_2|}_{\geq 3})$$

$$\neq (-2, 3)$$

Useful Facts about Vector Spaces

V vector space $k \in \mathbb{R}$, $\bar{u} \in V$.

(a) $0\bar{u} = \bar{0}$

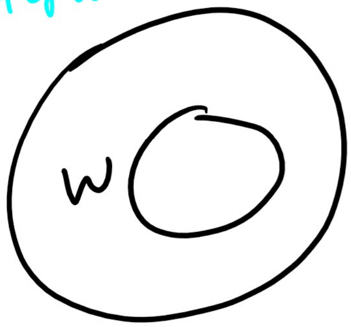
(b) $k\bar{0} = \bar{0}$

(c) $(-1)\bar{u} = -\bar{u}$

(d) If $k\bar{u} = \bar{0}$, then $k = 0$ or $\bar{u} = \bar{0}$.

For proofs of these facts, see Textbook p. 189, Theorem 4.1.1 or pp.6-7 of this document.

4.2 Subspaces



V - vector space

W subset of V

When is W (with <u>same</u> $+$ & scalar mult. as V) a vector space in its own right?

Since members of W are also members of V, we get A2, A3, A7, A8, A9, A10 for free. (Have a look at whats they each say.)

What about A1, A4, A5, A6 ?

↖ closure under scalar mult.

If A6 true for W

then $\underbrace{0\bar{u}}_{=\bar{0}}$ $\in W$ for any $\bar{u} \in W$ ($\bar{0}$ by Fact, since V is a vector space)

& $\underbrace{(-1)\bar{u}}_{=-\bar{u}} \in W$ ($-\bar{u}$ by Fact, since V is a vector space)

By Facts above applied to V, (where now indicated in light blue) $\bar{0}, -\bar{u} \in W$ so

This up here says exactly that $\bar{0} \in W$ & $-\bar{u} \in W$

(A4), (A5) for free. (almost) ↳ (if we have (A6) already).

So to check W is a vector space (& we say W is a subspace of V) then it's enough to check (A1) & (A6) (& W not empty).

Test for Subspaces If V is a vector space and W is a subset of V with the same operations of addition & scalar multiplication on objects as V has, then W is a subspace of V if

1. W is not empty;
2. W is closed under addition (i.e. (A1) true about W);
3. W is closed under scalar multiplication (i.e. (A6) true about W).

So if you have a collection of objects W with an addition operation & a scalar multiplication operation, but you can see that W is part of a bigger collection V which is a vector space when you use the same addition operation & scalar multiplication operation, then you can change the question "Is W a vector space?"

into "Is W non-empty? Is W closed under addition & scalar multiplication?" — instead of having to check 10+ things, you only have to check 3 !

Example proofs of Facts using Axioms

V vector space

- see textbook for (a), (c) (Theorem 4.1.1 p. 189).

(b) $k\bar{0} = \bar{0}$, for $k \in \mathbb{R}$.

$k\bar{0} = k(\bar{0}+\bar{0})$ (A4) — $\bar{0}+\bar{u}=\bar{u}$ for any $\bar{u}\in V$, so in particular $\bar{u}=\bar{0}$.

$k\bar{0} = k\bar{0}+k\bar{0}$ (A7)

So $\underbrace{k\bar{0}+(-(k\bar{0}))} = \underbrace{(k\bar{0}+k\bar{0})+(-(k\bar{0}))}$ (add $-(k\bar{0})$ to both sides)

$= \bar{0}$ (A5)

$= k\bar{0}+(k\bar{0}+(-(k\bar{0})))$ (A3)

$= k\bar{0}+\bar{0}$ (A5)

$= k\bar{0}$ (A4)

i.e. $\bar{0} = k\bar{0}$ as required.

(d) If $k\bar{u} = \bar{0}$, then either $k = 0$ or $\bar{u} = \bar{0}$.

Suppose $k\bar{u} = \bar{0}$. If $k \neq 0$, then we can consider

$$\left(\frac{1}{k}\right)k\bar{u} = \left(\frac{1}{k}\right)\bar{0}$$
$$= \bar{0} \quad \text{by (b)}$$

But of course $\left(\frac{1}{k}\right)k\bar{u} = \left(\frac{1}{k}k\right)\bar{u}$ (A9)

$$= 1\bar{u}$$
$$= \bar{u} \quad \text{(A10)}$$

i.e. $\bar{u} = \bar{0}$.

So either $k = 0$ or $\bar{u} = \bar{0}$ (& perhaps both).