1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

$\binom{WS\ 19}{C03}$ Lecture 27

Last Time

Subspaces of Vector Spaces

V : vector space with addition & scalar multiplication ops.

W : subset of V with <u>same</u> addition & scalar mult. ops.

Test for Subspaces If W, V are as above, then W is a subspace of V (i.e. itself a vector space with same ops. as V)

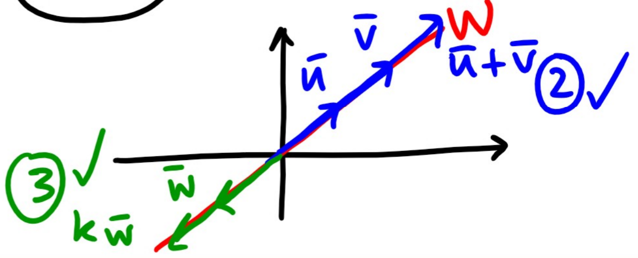
if
1. W is not empty;
2. W is closed under addition;
3. W is closed under scalar multiplication.

Examples SS# & Non-Examples NSS#

For any V vector space we have 2 "trivial" subspaces : SS1 $W = V$ SS2 $W = \{\bar{0}\}$

the zero vector in V given by Ax. 4.

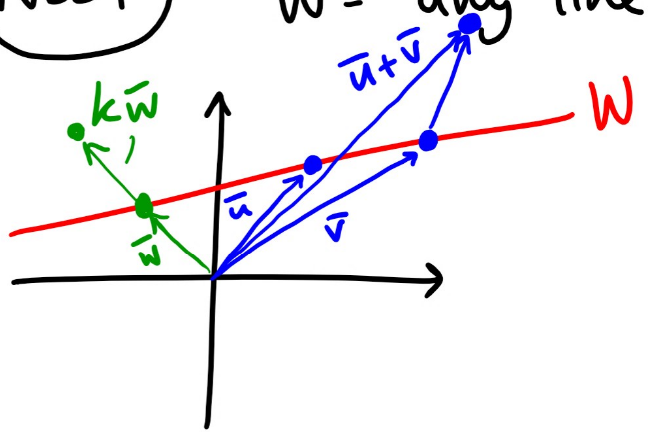
SS3 W = A line through the origin in $\mathbb{R}^2$ V



① $\bar{0} \in$ line ✓

Similarly W = a line or a plane through origin in $\mathbb{R}^3 = V$

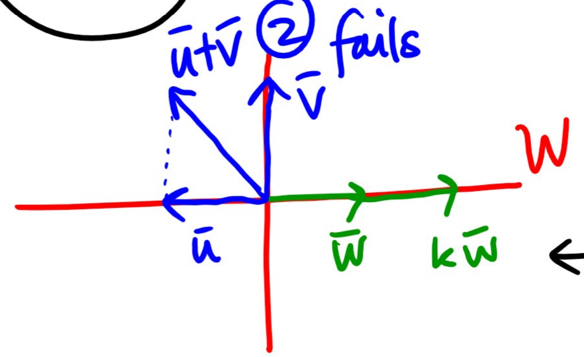
NSS 1 W = any line NOT through the origin in $\mathbb{R}^2 = V$



① ok pick a point on line ✓

②, ③ fail.

NSS 2 W = The x-axis & y-axis together in $\mathbb{R}^2 = V$



① $\bar{0} \in W$ ✓

③ ✓ ② ✗.

NSS3 W = Vectors (a,b,c) with $ab = c$. $V = \mathbb{R}^3$

✓ ① $(2,3,6)$ or $(0,0,0)$ etc...

③ Take $(a,b,c) \in W$ & $k \in \mathbb{R}$.

Look at $k(a,b,c) = (ka, kb, kc)$ ← Is this in W?

i.e. is $(ka)(kb) = kc$

Is $k^2ab = kc$?

Is $k^2c = kc$?

To cook up an example of failure make sure $c \neq 0$,
$k \neq 0, 1$

e.g. $5(2,3,6) = (10,15,30)$

$10 \times 15 \neq 30$.

② fails as well in a similar way. → see end of document.

(SS4) W = Diagonal matrices in $\boxed{M_{22}(\mathbb{R})} = V$

$\nwarrow$ 2×2

(Also works for any $M_{nn}(\mathbb{R})$)

① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ... ✓

② If $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ & $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$,

$$A + B = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix} \quad ✓$$

③ If $k \in \mathbb{R}$, $kA = \begin{bmatrix} ka_1 & 0 \\ 0 & ka_2 \end{bmatrix}$ ✓

The same idea works for W = UT matrices
or W = LT matrices

But not W = triangular matrices

e.g. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 8 \end{bmatrix} \leftarrow$ NOT Δ.

SS5 $W = P$, all polynomials, is a subspace of $F(-\infty, \infty)$

all $f: (-\infty, \infty) \to \mathbb{R}$

SS6 $W = P_d$, polys of degree $\leq d$, is subspace of P & also of $F(-\infty, \infty)$

For more spaces of functions see textbook pp. 194–195.

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3 really important types of subspaces

<u>Definition</u> If V is a vector space & $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ is a collection of "vectors" in V, then

$$\text{Span}(S) = \{\text{all linear combinations of } \bar{v}_1, \ldots, \bar{v}_r\}$$

is a subspace of V.

all vectors that look like $k_1\bar{v}_1 + \ldots + k_r\bar{v}_r$ for some choice of scalars k_i

Sometimes called "the subspace generated by S"

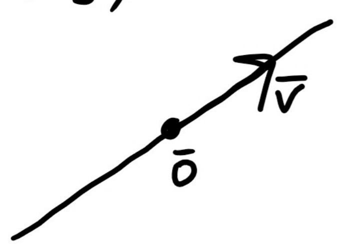
It is the "smallest" subspace of V containing all $\bar{v}_i$ in S.

SS7 In $\mathbb{R}^n$, $\operatorname{span}(\{\bar{e}_1, \ldots, \bar{e}_n\}) = \mathbb{R}^n$

(Any $\bar{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ can be written

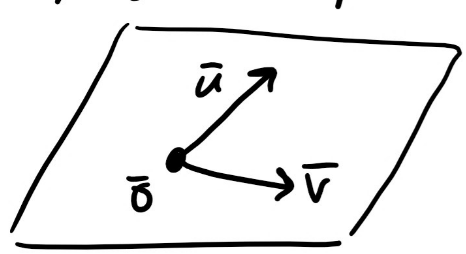
$= v_1\bar{e}_1 + \ldots + v_n\bar{e}_n$.

SS8 If $\bar{v} \in \mathbb{R}^n$, then $\operatorname{span}(\{\bar{v}\})$ is line through origin, direction $\bar{v}$



(see: parametric eq. of line)

If $\bar{u}, \bar{v} \in \mathbb{R}^n$, then $\operatorname{span}(\{\bar{u}, \bar{v}\})$ is plane through origin parallel to $\bar{u}, \bar{v}$



(see: param. eq. of plane)

(as long as $\bar{u}, \bar{v}$ NOT colinear)

SS9 All polynomials of degree at most 3 look like $a_0 + a_1x + a_2x^2 + a_3x^3$ ← P_3

$$a_0(1)$$

i.e. $\text{span}(\{1, x, x^2, x^3\}) = P_3$.

<u>Question</u> Does $\{1, x - x^2, x^3, 1 + x^2\}$ span P_3 too?

i.e. can we write every poly. $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

as $b_0(1) + b_1(x - x^2) + b_2 x^3 + b_3(1 + x^2)$?

i.e. can we find b_0, b_1, b_2, b_3 in terms of a_0, a_1, a_2, a_3.

$$\underbrace{(b_0 + b_3)}_{=a_0} + \underbrace{b_1}_{=a_1} x + \underbrace{(b_3 - b_1)}_{=a_2} x^2 + \underbrace{b_2}_{a_3} x^3$$

Question is: can we solve this linear system?

$$b_1 = a_1$$
$$b_2 = a_3$$

$$b_3 - b_1 = a_2$$
$$b_3 = a_2 + b_1 = a_2 + a_1$$

$$b_0 + b_3 = a_0$$
$$b_0 = a_0 - b_3 = a_0 - (a_2 + a_1).$$

So yes, we can & so $\{1, x - x^2, x^3, 1 + x^2\}$ also spans P_3.

<u>Question</u> What about $\{1, x+x^2, x^3\}$?

Try to write $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ as

$$= c_0(1) + c_1(x+x^2) + c_2 x^3$$

& get
$$a_0 = c_0$$
$$\left.\begin{array}{l} a_1 = c_1 \\ a_2 = c_1 \end{array}\right\}$$ Oh dear. If $a_1 \neq a_2$, then the answer will be no, there is no solution
$$a_3 = c_2$$

e.g. $p(x) = 3 + 5x + 7x^2 - 2x^3$

$c_1 = 5 \neq 7 = c_1$ problem!

But is the problem just that we didn't have "enough" vectors? (i.e. $\{1, x+x^2, x^3\}$ has 3 functions, whereas $\{1, x, x^2, x^3\}$ and $\{1, x-x^2, x^3, 1+x^2\}$ have 4.)

No. Try $\{1, x+x^2, x^3-2x^2-1, x-x^2+x^3\}$ Write:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = d_0 + d_1(x+x^2) + d_2(x^3-2x^2-1) + d_3(x-x^2+x^3)$$
$$= (d_0 - d_2) + (d_1+d_3)x + (d_1 - 2d_2 - d_3)x^2 + (d_2+d_3)x^3$$

$\Rightarrow a_0 = d_0 - d_2$

$a_1 = d_1 + d_3 \Rightarrow d_1 = a_1 - d_3$

$a_2 = d_1 - 2d_2 - d_3 \longrightarrow a_2 = (a_1 - d_3) - 2(a_3 - d_3) - d_3 = a_1 - 2a_3$

$a_3 = d_2 + d_3 \Rightarrow d_2 = a_3 - d_3$

not nec. true.

In the example NSS3 above, we said 2 (closure under addition) also fails. Why? $W = \{(a,b,c) \text{ with } ab = c\} \subseteq \mathbb{R}^3$.

Take (a_1, b_1, c_1) and (a_2, b_2, c_2) in W i.e. $a_1 b_1 = c_1$, $a_2 b_2 = c_2$

Then look at $(a_1, b_1, c_1) + (a_2, b_2, c_2)$

$$= (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

Does $(a_1 + a_2)(b_1 + b_2) = c_1 + c_2$?

i.e. $a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1 = c_1 + c_2$?

i.e. $c_1 + c_2 + a_1 b_2 + a_2 b_1 = c_1 + c_2$?

Not if $a_1 b_2 + a_2 b_1 \neq 0$. So cook up an example where we have that eg take $a_1 = 1$, $b_1 = 1$ (so $c_1 = 1$) and $a_2 = 1$; make sure $b_2 \neq -1$ & we'll be OK e.g. $b_2 = 1$ (so $c_2 = 1$).

Then note $(\underbrace{1, 1}_{1 \times 1}, \underbrace{1}_{= 1}) + (\underbrace{1, 1}_{1 \times 1}, \underbrace{1}_{= 1}) = (\underbrace{2, 2}_{2 \times 2}, \underbrace{2}_{\neq 2})$.

So 2 fails.