1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 28

Yesterday

The <u>span</u> of $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ ← a bunch of vectors in vector space V

is

$$\text{Span}(S) = \{\underbrace{k_1\bar{v}_1 + \dots + k_r\bar{v}_r}_{\text{all linear combinations of } \bar{v}_1, \dots, \bar{v}_r.} \text{ for all choices of scalars } k_1, \dots, k_r\}$$

→ It is a <u>subspace</u> of V (though <u>in general S won't be</u>).

e.g. • line in $\mathbb{R}^n$ = $\text{Span}(\{\bar{v}\})$ • plane in $\mathbb{R}^n$ = $\text{Span}(\{\bar{u}, \bar{v}\})$ ($\bar{u}, \bar{v}$ NOT colinear)

• $P_d = \text{Span}(\{1, x, \dots, x^d\})$

<u>Example #2</u> of an important type of subspace

Take an $m \times n$ matix A

<u>Def^n</u> The set of solutions $\bar{x}$ to $A\bar{x} = \bar{0}$ is a subspace of $\mathbb{R}^n$ → called the <u>null space</u> of A or the <u>kernel of A</u> (or the <u>solution space to $A\bar{x} = \bar{0}$</u>).

Test: ① $A\bar{0} = \bar{0}$ ✓

② If $\bar{x}_1, \bar{x}_2$ are solutions, then $A(\bar{x}_1 + \bar{x}_2)$

$= A\bar{x}_1 + A\bar{x}_2$

$= \bar{0} + \bar{0} = \bar{0}$ ✓

③ If $\bar{x}$ is a soln, k scalar

$A(k\bar{x}) = k(A\bar{x}) = k\bar{0} = \bar{0}$ ✓

SS10 $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & 2 \end{bmatrix}$ has solutions to $A\bar{x} = \bar{0}$ given by RREF of

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$z = t$

$x = -2t,\ y = -t,$

Solutions : $\bar{x} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} t$ i.e.

null space of A is $\mathrm{Span}\left(\left\{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right\}\right)$

Example #3 of important type of subspace

<u>Def^n</u> A $n \times n$ matrix

The <u>λ-eigenspace of A</u> is $E_\lambda = \{\bar{x} \text{ with } A\bar{x} = \lambda\bar{x}\}$, a subspace of $\mathbb{R}^n$

(if λ is an eigenvalue of A)

then this actually has meaningful content, though the definition makes sense for any λ, and E_λ always contains $\bar{0}$...

① $A\bar{0} = \bar{0} = \lambda\bar{0}$ ✓

② If $A\bar{x}_1 = \lambda\bar{x}_1$, $A\bar{x}_2 = \lambda\bar{x}_2$, then ...

Did you try? $A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \lambda\bar{x}_1 + \lambda\bar{x}_2 = \lambda(\bar{x}_1 + \bar{x}_2)$ ✓

③ If $A\bar{x} = \lambda\bar{x}$, k scalar, then ...

$A(k\bar{x}) = k(A\bar{x}) = k(\lambda\bar{x}) = \lambda(k\bar{x})$ ✓

Example SS11 In Lecture 15 we found $\lambda = 5$ is an eigenvalue of $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix}$

We said a <u>basis</u> for $\underbrace{\text{eigenvectors of } \lambda = 5 \text{ together with } \bar{0}}_{\text{AKA eigenspace of } \lambda=5,\ E_5}$ was

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$$

i.e. $E_5 = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\} \right)$

<u>Notice</u> We find eigenvectors by solving equations

$$(A - \lambda I)\bar{x} = \bar{0}.$$

So eigenspaces are always null spaces.

<u>Question</u> If W is a subspace of V (vector space) can we write down a list S of vectors in V so that $W = \text{Span}(S)$?

Yeah sure - write down every vector in W! But surely we can do better?

(We'd say "S is a <u>spanning set</u> for W")

How many vectors in S do we need? (few = good)

<u>Observation</u> (Thm 4.2.6)

If $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ and $T = \{\bar{w}_1, \dots, \bar{w}_m\}$

are vectors in U with $\mathrm{Span}(S) = \mathrm{span}(T)$

then all $\bar{v}_i \in \mathrm{span}(T)$

all $\bar{w}_j \in \mathrm{Span}(S)$.

[And the Theorem says the reverse is true too — if $\bar{v}_i \in \mathrm{span}(T)$ for all i & $\bar{w}_j \in \mathrm{span}(S)$ for all j, then $\mathrm{span}(S) = \mathrm{span}(T)$]

<u>Example</u> $\mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right\}$

$= \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

in Span

$= \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$

UP TO HERE FOR TEST #2

4.3 Linear Independence ("No redundancy" in spanning)

<u>Def^n</u> $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ is <u>linearly independent</u> if the only choice of scalars $k_1, \ldots, k_r$ with $k_1\bar{v}_1 + \ldots + k_r\bar{v}_r = \bar{0}$ is

$0 = k_1 = k_2 = \ldots = k_r$.

(S is linearly dependent otherwise.)

↳ i.e. at least one of the vectors in S can be written as a linear combination of the others. e.g. $k_1\bar{v}_1 + k_2\bar{v}_2 + \ldots + k_r\bar{v}_r = \bar{0}$ with $k_2 \neq 0 \rightsquigarrow$ then we could write $\bar{v}_2 = -\frac{1}{k_2}(k_1\bar{v}_1 + k_3\bar{v}_3 + \ldots + k_r\bar{v}_r)$

$= -\frac{k_1}{k_2}\bar{v}_1 - \frac{k_3}{k_2}\bar{v}_3 - \ldots - \frac{k_r}{k_2}\bar{v}_r$, a linear comb. of $\bar{v}_1, \bar{v}_3, \ldots, \bar{v}_r$.

Example

$\{\bar{e}_1, \ldots, \bar{e}_n\} \subseteq \mathbb{R}^n$ is linearly independent:

$$\bar{0} = k_1\bar{e}_1 + k_2\bar{e}_2 + \ldots + k_n\bar{e}_n = (k_1, k_2, \ldots, k_n)$$

$\Rightarrow k_i = 0$ for all i

Example Is $S = \{(1,0,0,1,1), (3,1,-1,0,2), (0,-2,1,3,0)\}$ linearly dependent?

Solution Set $\bar{0} = k_1(1,0,0,1,1) + k_2(3,1,-1,0,2) + k_3(0,-2,1,3,0)$

The question is, can we find k_1, k_2, k_3 not all zero to make this true?

i.e. is there a non-zero solution to the system with augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \xrightarrow[\text{Row Reduce}]{?} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i.e. $k_1 = k_2 = k_3 = 0$.

So no, no redundancy, S not linearly dependent (in fact S linearly independent).

<u>Example</u> $\{1, x, x^2, \dots, x^d\} \subseteq P_d$

↳ is linearly independent.

If $k_0 + k_1 x + k_2 x^2 + \dots + k_d x^d = 0$, how to check $k_i = 0$ for all i?

Plug in $x = 0$: $k_0 = 0$.

Differentiate! $k_1 + 2k_2 x + \dots + d k_d x^{d-1} = 0$

Plug in $x = 0$: $k_1 = 0$

$\vdots$ etc.

The essential point is that knowing that $\{1, x, \dots, x^d\}$ is linearly independent means we can compare polys by comparing coefficients.

<u>Example</u> Is $S = \{1 + x^3, x - x^2, x^2 - 1, x + x^3\}$ linearly independent (in P_3)?

Spend 5 minutes thinking about this. What do you think? We'll look at it on Thursday!