1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 29

Last Time   Linear Independence

$S = \{\bar{v}_1, \dots, \bar{v}_r\}$ is linearly independent if the only choice of scalars $k_1, \dots, k_r$ with $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$, is $k_i = 0$ for all i.

↳ This says "There's no way to write any vector $\bar{v}_i$ from S as a "non-trivial" linear combination of (some of) the other vectors in S."

To check: set $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$ and solve for $k_1, \dots, k_r$.

Recall: $\{1, x, x^2, \dots, x^d\}$ is linearly independent in P_d

Example: Is $S = \{1+x^3, x-x^2, x^2-1, x+x^3\}$ linearly independent?

coefficients here in blue:

Solution: Well $1(1+x^3) + 1(x-x^2) + 1(x^2-1) + (-1)(x+x^3) = 0$

So answer is no. → because we found a collection of coefficients, not all zero to make RHS = 0

Or: Write down $k_1(1+x^3) + k_2(x-x^2) + k_3(x^2-1) + k_4(x+x^3) = 0$

↑ If you couldn't spot an answer immediately

& solve for k_1, k_2, k_3, k_4.

i.e. $(k_1 - k_3) + (k_2 + k_4)x + (-k_2 + k_3)x^2 + (k_1 + k_4)x^3 = 0$

So we have solve LEs:

$$\begin{aligned} k_1 \quad\quad - k_3 \quad\quad &= 0 \\ k_2 \quad\quad + k_4 &= 0 \\ -k_2 + k_3 \quad\quad &= 0 \\ k_1 \quad\quad \cancel{k_3} + k_4 &= 0 \end{aligned}$$

We know $k_1 = k_2 = k_3 = k_4 = 0$ is a solution, so here, because we found a solution above, we know ≥ 2 solns hence

i.e. row reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \dashrightarrow$$

(Expect ∞ many solutions)

(only options for a system of LEs: 0, 1 or ∞-many solutions)

Example Show that $\{\sin(x), \cos(x)\}$ is linearly independent in $F(-\infty, \infty)$.

Solution Write down $k_1 \sin(x) + k_2 \cos(x) = 0$.

Here can be crafty e.g. plug in $x = 0$

$$k_1 \underbrace{\sin(0)}_{=0} + k_2 \underbrace{\cos(0)}_{=1} = 0$$

$$\Rightarrow k_2 = 0$$

e.g. $x = \frac{\pi}{2}$

$$k_1 \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} + k_2 \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} = 0$$

$$\Rightarrow k_1 = 0.$$

In general, if $\{f_1(x), \ldots, f_n(x)\}$ is a set of $(n-1)$ times differentiable functions, and the set is linearly dependent, then:

there are $k_1, \dots, k_n$ NOT all zero with

$$k_1 f_1(x) + k_2 f_2(x) + \dots + k_n f_n(x) = 0 \quad \text{for all } x$$

differentiate: $k_1 f_1'(x) + k_2 f_2'(x) + \dots + k_n f_n'(x) = 0$ for all x

& again $\vdots$ $\vdots$

$\vdots$

$$k_1 f_1^{(n-1)}(x) + \dots + k_n f_n^{(n-1)}(x) = 0 \text{ for all } x$$

n equations in n unknowns $k_1, \dots, k_n$

We can write it

$$\begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ \vdots & & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for all } x$$

Because this system has a non-trivial solution, for every x

$$\underbrace{\det \begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ \vdots & & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}}_{} = 0 \quad \text{for every } x.$$

$= W(x)$ called the Wronskian of $\{f_1, \dots, f_n\}$.

Fact If $f_1(x), \ldots, f_n(x)$ are $n-1$ times differentiable, then $\{f_1(x), \ldots, f_n(x)\}$ is linearly <u>in</u>dependent if $W(x) \neq 0$ for some x.

Example $\{1, e^x, \sin(x)\}$ is linearly independent.

Solution

$$W(x) = \det \begin{bmatrix} 1 & e^x & \sin(x) \\ 0 & e^x & \cos(x) \\ 0 & e^x & -\sin(x) \end{bmatrix} = 1 \begin{vmatrix} e^x & \cos(x) \\ e^x & -\sin(x) \end{vmatrix}$$

$$= e^x(-\sin(x)) - e^x(\cos(x))$$

$$= -e^x(\sin(x) + \cos(x))$$

And e.g. $W(0) = -1 \neq 0$ so set is linearly independent.

(We found a value of x for which $W(x) \neq 0$.)

Facts (1) If a finite set $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ contains $\bar{0}$, then S is linearly dependent.

Write down $k_1\bar{v}_1 + \ldots + k_n\bar{0} + \ldots + k_r\bar{v}_r = \bar{0}$

Then a solution for the k_is is $k_i = 0$ for $i \neq n$, $k_n = \text{anything} \neq 0$

(2) If $S = \{\bar{v}\}$, then S is linearly independent unless $\bar{v} = \bar{0}$ ($k\bar{v} = \bar{0} \Rightarrow k = 0$ unless $\bar{v} = \bar{0}$).

But $k\bar{0} = \bar{0}$ for any k so $\{\bar{0}\}$ is lin. depend. (see (1)).

(3) If $S = \{\bar{u}, \bar{v}\}$, then S is linearly independent as long as $\bar{u}, \bar{v}$ are NOT colinear.

(4) If $S = \{\bar{v}_1, \ldots, \bar{v}_r\} \subseteq \mathbb{R}^n$, then if $r > n$ then S is linearly dependent. (see textbook)

4.4 Coordinates & Bases ← plural of basis not base

If we have $(1, -3, 2)$ in $\mathbb{R}^3$ the coordinates are given relative to x, y, z -axes i.e. directions along positive x, y, z -axes

$\bar{e}_1, \bar{e}_2, \bar{e}_3$ lie in these three directions

Shift perspective $(1, -3, 2)$

$$= \underline{1}\,\bar{e}_1 \underline{-3}\,\bar{e}_2 \underline{+2}\,\bar{e}_3$$

coordinates are the coefficients of $\bar{e}_1, \bar{e}_2, \bar{e}_3$ in order.