1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 3

Last time

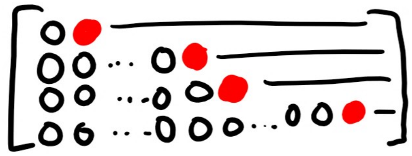
left-most non-zero entry $[0 \dots 0 \bullet \dots]$

REF / RREF:

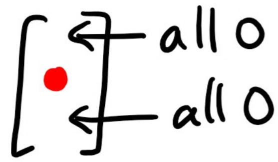
(1) The $\boxed{\text{leading entry}}$ in every non-zero row is 1. $\bullet = 1$

(2) Every zero row: $[0\ 0 \dots 0]$ is at the bottom.

(3) Every leading entry is to the right of the leading entries in the rows above (i.e. they stagger right & down:



(4) If a column has a leading entry, all its other entries are 0:



(R)REF = (Reduced) Row Echelon Form.

Example

$$\begin{array}{c}
\begin{array}{ccccc} w & x & y & z & \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 3 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}$$

✓ REF
✓ RREF

pivot columns

Corresponds to:
$$\begin{aligned} w + x \quad + 3z &= -5 \\ y - z &= -2 \end{aligned}$$

x & z free variables

Set the other variables (w, y) in terms of the free ones (x, z):

$$\begin{aligned} w &= -5 - x - 3z \\ y &= -2 + z \end{aligned}$$

Now set free variables to equal parameters e.g. $x = s$, $z = t$

We get: $w = -5 - s - 3t$
$x = s$
$y = -2 + t$
$z = t$

i.e. our solutions are given by

$(-5-s-3t, s, -2+t, t)$.

We know:

System of L.E.s $\longrightarrow$ (augmented) matrix

RREF matrix $\longrightarrow$ solution(s)

GOAL

How? <u>Elementary Row Operations</u>

(1) Scale a row by a non-zero constant.
(2) Add a multiple of one row to another.
(3) Swap any two rows over.

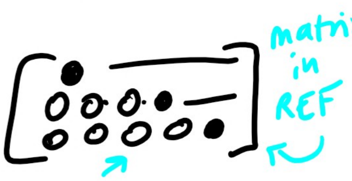
[1.2] Gauss – Jordan Elimination

To solve a system of L.E.s:

Step 1 Write down the augmented matrix of the system.

Step 2 Use elementary row operations to turn the augmented matrix into a matrix in RREF.

Step 3 Read off the solution(s) from the RREF matrix.

Remark 

If we stop part way through step 2 at a matrix in REF, this is called Gaussian Elimination.

Then to find solution(s) we use "back substitution"

— implicitly we did this with $\begin{cases} x+3y=6 \\ 2x-y=1 \end{cases}$

Solution is $\left(\frac{9}{7}, \frac{11}{7}\right)$. We got this from changing system to

$$\begin{cases} x+3y=6 \\ \quad\; y = 11/7 \end{cases}$$

so

$$x = 6-3y = 9/7.$$

Original system:

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1}$$

matrix

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 11/7 \end{bmatrix} \leftarrow \text{REF}$$

Elimination Procedure

Solve

$$\begin{aligned} 3y - z &= -2 \\ 2x \quad\; + 2z &= 1 \\ x + y \quad\; &= 5. \end{aligned}$$

Solution

Step 1 Augmented matrix:

$$\begin{bmatrix} 0 & 3 & -1 & -2 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

Step 2 G-J Elim.

Stage I (Gaussian Elim.) $\longrightarrow$ REF

(1) Find left-most non-zero column & make sure its top entry is NOT 0 (swap rows)

$\downarrow$ Swap R_1 and R_2

$$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 3 & -1 & -2 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

(2) Multiply through top row so leading entry is 1

$$\downarrow R_1 \to \tfrac{1}{2} R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 3 & -1 & -2 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

(3) "Kill off" rest of row 1's leading entry column (set to 0) by adding multiples of row 1 to other rows:

$$\downarrow R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 1/2 \\ \hline 0 & 3 & -1 & -2 \\ 0 & 1 & -1 & 9/2 \end{bmatrix}$$

(4) Cover up top row & repeat from (1):

$$\begin{bmatrix} 0 & 3 & -1 & -2 \\ 0 & 1 & -1 & 9/2 \end{bmatrix}$$

(1)

$$\xrightarrow[(2)]{R_1 \to \frac{1}{3} R_1}$$

$$\begin{bmatrix} 0 & 1 & -1/3 & -2/3 \\ 0 & 1 & -1 & 9/2 \end{bmatrix}$$

$$\xrightarrow[(3)]{R_2 \to R_2 - R_1}$$

$$\begin{bmatrix} 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & -2/3 & 31/6 \end{bmatrix} \xrightarrow{(4)} \begin{bmatrix} 0 & 0 & -\frac{2}{3} & \frac{31}{6} \end{bmatrix}$$

$$\uparrow \frac{9}{2} + \frac{2}{3}$$

$$\downarrow (1)\ R_1 \to -\tfrac{3}{2} R_1$$

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{31}{4} \end{bmatrix}$$

Stop after last row and uncover rows:

$$\text{REF} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 1 & -31/4 \end{bmatrix}$$

Stage II (REF → RREF)

$$\downarrow \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + \frac{1}{3} R_3 \end{array}$$

(1) Add multiples of last non-zero row to the rows above to "kill off" entries above the leading 1 in the last row.

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 33/4 \\ 0 & 1 & 0 & -13/4 \\ 0 & 0 & 1 & -31/4 \end{array}\right]$$

↑ RREF

(2) If needed repeat previous step with 2nd-to-last non-zero row, 3rd-to-last etc. ... moving from right to left (bottom to top).

We end up here with RREF.

Step 3 Now read off the solution(s):

$$x = \frac{33}{4}$$

$$y = -\frac{13}{4}$$

$$z = -\frac{31}{4}.$$