1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 30

Recall A collection $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ of vectors in V

- spans V (or a subspace W) if every vector $\bar{v}$ in V (resp. W) can be written as $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{v}$.
- is linearly independent if the only choice of scalars $k_1, \dots, k_r$ with $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$ is $k_1 = \dots = k_r = 0$.

Today Coordinates e.g. $(3,-1,2) = \underline{3}\bar{e}_1 + (\underline{-1})\bar{e}_2 + \underline{2}\bar{e}_3$

Definitions A set S of vectors in a vector space V is a basis for V if
(1) $\text{span}(S) = V$
(2) S is linearly independent

If $S = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a finite basis for V, and $\bar{v}$ in V is written $\bar{v} = c_1\bar{v}_1 + \dots + c_n\bar{v}_n$ then $c_1, \dots, c_n$ are called the coordinates of $\bar{v}$ relative to S (or wrt S) & $(\bar{v})_S = (c_1, \dots, c_n) \in \mathbb{R}^n$ is the coordinate vector of $\bar{v}$ relative to S.

If $\bar{v} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n = d_1\bar{v}_1 + \ldots + d_n\bar{v}_n$

i.e. if we potentially had two coordinate vectors for $\bar{v}$ wrt S

$\Rightarrow (c_1 - d_1)\bar{v}_1 + \ldots + (c_n - d_n)\bar{v}_n = \bar{0}$

$\Rightarrow c_i - d_i = 0$ for all i as S is linearly independent.

<u>Example</u> In $\mathbb{R}^n$ the <u>standard basis</u> is

$$\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}.$$

$\bar{v} = (v_1, \ldots, v_n) = v_1\bar{e}_1 + \ldots + v_n\bar{e}_n$

$v_1, \ldots, v_n$ are the coordinates of $\bar{v}$ relative to $\{\bar{e}_1, \ldots, \bar{e}_n\}$.

<u>Example</u> In P_d, $\{1, x, x^2, \ldots, x^d\}$ is the standard basis for P_d.

- We already showed span $= P_d$ & linear independence
- A polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_dx^d$ has coordinates $a_0, a_1, a_2, \ldots, a_d$ wrt the standard basis.

<u>Example</u> In $M_{mn}(\mathbb{R})$ there is a standard basis: the $m.n$-many $m \times n$ matrices with a 1 in one

entry & rest zeros

e.g. for $M_{22}(\mathbb{R})$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Spans : take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \dots$

Lin. Ind. : exerise!

<u>Example</u> Show that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

<u>Solution</u> A linear comb. of these vectors:

$$c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\bar{c}} = A\bar{c}$$

linear comb.

So : • "set spans $\mathbb{R}^3$" is the same as
"for every $\bar{b} \in \mathbb{R}^3$, we can write $\bar{b}$ as $A\bar{c}$"
i.e. "for every $\bar{b}$, there is a solution $\bar{c} \in \mathbb{R}^3$ with $A\bar{c} = \bar{b}$"

- "set linearly independent" is the same as "the only solution for $\bar{c}$ to $A\bar{c} = \bar{0}$ is $\bar{c} = \bar{0}$"

Both of these are true exactly when A is invertible (notice: A square)

So check $\det(A) = 0$ or not:

$$\det(A) = \begin{vmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -4 \neq 0$$ so A invertible so the set is a basis.

<u>Example</u> Show that $S = \{1+x,\ x+x^2,\ x^2\}$ is a basis for P_2 & find the coordinates of $5 - 3x + x^2$ relative to S.

<u>Solution</u> Linear comb. of vectors in S:

$$c_1(1+x) + c_2(x+x^2) + c_3x^2$$

Set this $= a_0 + a_1x + a_2x^2$

We get $\quad c_1 + (c_1+c_2)x + (c_2+c_3)x^2$

i.e. want c_1, c_2, c_3 with

$$\begin{aligned} c_1 &= a_0 \\ c_1 + c_2 &= a_1 \\ c_2 + c_3 &= a_2 \end{aligned}$$

In matrix form: $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\bar{c}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$

- "Span$(S) = P_2$" = "for any choice of a_0, a_1, a_2 we have a solution for $\bar{c}$"

- "S lin. independent" = "the only solution to $A\bar{c} = \bar{0}$ is $\bar{c} = \bar{0}$"

So again S is a basis iff A is invertible

$\det(A) = 1 \neq 0 \rightarrow A$ invertible.

To write $5 - 3x + x^2$ in terms of S,
we solve for $\bar{c}$ in $A\bar{c} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

i.e. row reduce $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 1 & 1 & 0 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 9 \end{array}\right]$

So $c_1 = 5, c_2 = -8, c_3 = 9$

$$(5 - 3x + x^2)_S = (5, -8, 9) \in \mathbb{R}^3.$$

$$5 - 3x + x^2 = 5(1+x) - 8(x+x^2) + 9x^2$$

More on bases later but first:

6.3 Gram - Schmidt Process

<u>Example</u> Find the coordinate vector of $\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}$ relative to the basis

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

Method so far: row reduce

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 1 & -2 & 1 & 7 \\ 0 & 1 & 4 & 7 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So $\left(\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix} \right)_S = (3, -1, 2)$

i.e. $\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$

But notice what happens when we take dot products of the vectors in S . . .