1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 31

Yesterday Bases & Coordinates

- $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ is a **basis** for V if (1) $\text{span}(S) = V$
  (2) S is linearly independent
- If S is a basis for V and $\bar{v} = c_1\bar{v}_1 + \dots + c_r\bar{v}_r$, then $(\bar{v})_S = (c_1, \dots, c_n)$ is the vector of coordinates of $\bar{v}$ w.r.t. S

e.g. $\{\bar{e}_1, \dots, \bar{e}_n\}$ in $\mathbb{R}^n$; $\{1, x, \dots, x^d\}$ in P_d ; also ...

standard bases

Example (again) Find the coord. vector for $\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}$

wrt $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}$.

basis

Yesterday's method: solve $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 1 & -2 & 1 & 7 \\ 0 & 1 & 4 & 7 \end{array}\right]$

row reduction $\rightsquigarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

So $\left(\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}\right)_S = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

Notice

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 0$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = 0$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = 0$

6.3 Gram-Schmidt Process

When the vectors in a set $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ satisfy $\bar{v}_j \cdot \bar{v}_i = 0$ for $j \neq i$, we call S an <u>orthogonal set</u> (or <u>orthogonal basis</u> if S is a basis)

If all $\bar{v}_i$ are also unit vectors, we say <u>orthonormal</u> set/basis

If S is an orthogonal basis for V & $\bar{v} \in V$

write $\bar{v} = c_1\bar{v}_1 + \dots + c_r\bar{v}_r$.

For each $\bar{v}_i$

$$\begin{aligned}\bar{v} \cdot \bar{v}_i &= (c_1\bar{v}_1 + \dots + c_r\bar{v}_r) \cdot \bar{v}_i \\ &= c_1\bar{v}_1 \cdot \bar{v}_i + \dots + c_r\bar{v}_r \cdot \bar{v}_i \\ &= c_i(\bar{v}_i \cdot \bar{v}_i) = c_i \|\bar{v}_i\|^2\end{aligned}$$

$$\Rightarrow \quad c_i = \frac{\bar{v} \cdot \bar{v}_i}{\underbrace{\|\bar{v}_i\|^2}_{=1 \text{ if } S \text{ is an orthonormal basis.}}}$$

for each $i = 1, \dots, r$.

<u>Back to example</u>

Write $\underbrace{\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}}_{\bar{v}}$ in terms of basis $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_1}, \underbrace{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}_{\bar{v}_2}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}}_{\bar{v}_3} \right\}$

$$c_1 = \frac{\bar{v} \cdot \bar{v}_1}{\|\bar{v}_1\|^2} = \frac{-1+7}{1+1} = \frac{6}{2} = 3$$

$$c_2 = \frac{\bar{v} \cdot \bar{v}_2}{\|\bar{v}_2\|^2} = \frac{-2-14+7}{4+4+1} = \frac{-9}{9} = -1$$

$$c_3 = \frac{\bar{v} \cdot \bar{v}_3}{\|\bar{v}_3\|^2} = \frac{1+7+28}{1+1+16} = \frac{36}{18} = 2.$$

<u>Remember</u> this trick ONLY works when S is orthogonal !!!

Also :

<u>Fact</u> If $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ is an orthogonal set of non-zero vectors, then S is linearly independent.

Why? Set $\bar{0} = k_1 \bar{v}_1 + \dots + k_r \bar{v}_r$

Again, take dot product with each $\bar{v}_i$ in turn:

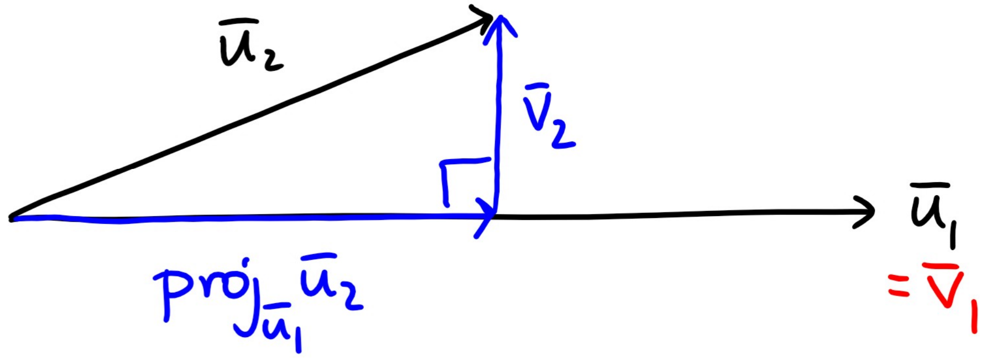
$$\begin{aligned} 0 = \bar{0} \cdot \bar{v}_i &= (k_1 \bar{v}_1 + \dots + k_r \bar{v}_r) \cdot \bar{v}_i \\ &= k_1 \bar{v}_1 \cdot \bar{v}_i + \dots + k_r \bar{v}_r \cdot \bar{v}_i \\ &= k_i \underbrace{\bar{v}_i \cdot \bar{v}_i}_{= \|\bar{v}_i\|^2 \neq 0} \Rightarrow k_i = 0. \end{aligned}$$

for each i.

<u>How to get an orthogonal basis from any basis</u>

In $\mathbb{R}^3$, $S = \{\bar{u}_1, \bar{u}_2\}$ is a basis for the plane that it spans (i.e. $W = \text{span}(\{\bar{u}_1, \bar{u}_2\})$)

as long as $\bar{u}_1, \bar{u}_2$ NOT colinear. (This says S linearly independent)



By Projection Theorem

$$\bar{u}_2 = \text{proj}_{\cancel{\bar{u}_1}\, \bar{v}_1} \bar{u}_2 + \bar{v}_2$$

<u>Notice</u>: $\text{span}\{\bar{v}_1, \bar{v}_2\} = \text{span}\{\bar{u}_1, \bar{u}_2\} = W$

AND $\{\bar{v}_1, \bar{v}_2\}$ is an orthogonal basis for W

where $\bar{v}_1 = \bar{u}_1$, $\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1} \bar{u}_2$

<u>Example</u> Find an orthogonal basis for the plane spanned by $\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\bar{u}_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}_2} \right\}$.

<u>Solution</u> $\bar{v}_1 = \bar{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

$$\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1} \bar{u}_2 = \bar{u}_2 - \frac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \underbrace{\frac{2-1+0}{1+1+9}}_{1/11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/11 \\ -1/11 \\ 3/11 \end{bmatrix} = \begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}$$

So answer: $\{\bar{v}_1, \bar{v}_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix} \right\}$.

Notice $\bar{v}_1 \cdot \bar{v}_2 = \frac{21}{11} - \frac{12}{11} - \frac{9}{11} = 0$.

<u>Gram - Schmidt Procedure</u> $\rightarrow$ above idea with maybe more vectors

$S = \{\bar{u}_1, \ldots, \bar{u}_r\}$ — basis for subspace W of $\mathbb{R}^n$

The following produces $\{\bar{v}_1, \ldots, \bar{v}_r\}$, orthogonal basis for W &

$$\text{span}\{\bar{u}_1, \ldots, \bar{u}_k\} = \text{span}\{\bar{v}_1, \ldots, \bar{v}_k\} \text{ for } 1 \leq k \leq r.$$

$$\bar{v}_1 = \bar{u}_1$$

Take away the part of $\bar{u}_2$ coming from $\bar{v}_1$ & get the part of $\bar{u}_2$ orthogonal to $\bar{v}_1$

$$\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1} \bar{u}_2 = \bar{u}_2 - \frac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1$$

Take away the parts of $\bar{u}_3$ coming from each of the vectors already on the new list $[\bar{v}_1, \bar{v}_2]$ & get the part of $\bar{u}_3$ orthogonal to $\bar{v}_1$ and $\bar{v}_2$.

$$\bar{v}_3 = \bar{u}_3 - \text{proj}_{\bar{v}_1} \bar{u}_3 - \text{proj}_{\bar{v}_2} \bar{u}_3$$

$$= \bar{u}_3 - \frac{\bar{u}_3 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1 - \frac{\bar{u}_3 \cdot \bar{v}_2}{\|\bar{v}_2\|^2} \bar{v}_2$$

$\vdots$

Take away the part of $\bar{u}_r$ coming from all the new vectors already on the list $[\bar{v}_1, \ldots, \bar{v}_{r-1}]$ and get the part of $\bar{u}_r$ orthogonal to all of $\bar{v}_1, \ldots, \bar{v}_{r-1}$.

$$\bar{v}_r = \bar{u}_r - \text{proj}_{\bar{v}_1} \bar{u}_r - \text{proj}_{\bar{v}_2} \bar{u}_r - \ldots - \text{proj}_{\bar{v}_{r-1}} \bar{u}_r$$

To get an <u>orthonormal</u> basis for W, 2 options:

1. Replace $\{\bar{v}_1, \dots, \bar{v}_r\}$ by $\left\{\frac{\bar{v}_1}{\|\bar{v}_1\|}, \dots, \frac{\bar{v}_r}{\|\bar{v}_r\|}\right\}$.

2. At every stage, normalize, and continue on, working with the normalized vector instead.