1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 32

Last Time Gram-Schmidt Process

How to get an orthogonal basis for subspace W in $\mathbb{R}^n$ from any basis for W.

Old basis : $\{\bar{u}_1, \ldots, \bar{u}_r\}$

New basis : $\{\bar{v}_1, \ldots, \bar{v}_r\}$ with $\operatorname{span}(\{\bar{u}_1, \ldots, \bar{u}_k\}) = \operatorname{span}(\{\bar{v}_1, \ldots, \bar{v}_k\})$ for $1 \le k \le r$

$$\hookrightarrow \bar{v}_1 = \bar{u}_1$$

$$\vdots$$

$$\bar{v}_n = \bar{u}_n - \overbrace{\frac{\bar{u}_n \cdot \bar{v}_1}{\|\bar{v}_1\|^2}\bar{v}_1}^{\operatorname{proj}_{\bar{v}_1}\bar{u}_n} - \overbrace{\frac{\bar{u}_n \cdot \bar{v}_2}{\|\bar{v}_2\|^2}\bar{v}_2}^{\operatorname{proj}_{\bar{v}_2}\bar{u}_n} - \cdots - \overbrace{\frac{\bar{u}_n \cdot \bar{v}_{n-1}}{\|\bar{v}_{n-1}\|^2}\bar{v}_{n-1}}^{\operatorname{proj}_{\bar{v}_{n-1}}\bar{u}_n}$$

$$\vdots$$

To get an orthonormal basis : $\left\{\frac{v_1}{\|v_1\|}, \ldots, \frac{v_r}{\|v_r\|}\right\}$

Example Find an orthonormal basis for

$$W = \operatorname{span}(\{\underbrace{(1,0,1)}_{\bar{u}_1}, \underbrace{(0,5,-2)}_{\bar{u}_2}, \underbrace{(4,-3,2)}_{\bar{u}_3}\})$$

Solution

$$\bar{v}_1 = \bar{u}_1$$

$$\bar{v}_1 = (1,0,1) \quad \rightsquigarrow \|(1,0,1)\| = \sqrt{1+1} = \sqrt{2}$$

$$\bar{v}_2 = \bar{u}_2 - \operatorname{proj}_{\bar{v}_1}\bar{u}_2 = (0,5,-2) - \frac{(0,5,-2)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1)$$

$$= (0,5,-2) - \frac{-2}{2}(1,0,1) = (1,5,-1)$$

$+$

$$\|(1,5,-1)\| = \sqrt{1+25+1} = \sqrt{27} = 3\sqrt{3}$$

$\bar{v}_3 = \bar{u}_3 - \text{proj}_{\bar{v}_1} \bar{u}_3 - \text{proj}_{\bar{v}_2} \bar{u}_3$

$= (4,-3,2) - \dfrac{(4,-3,2)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1) - \dfrac{(4,-3,2)\cdot(1,5,-1)}{\|(1,5,-1)\|^2}(1,5,-1)$

$= (4,-3,2) - \dfrac{6}{2}(1,0,1) - \dfrac{4-15-2}{27}(1,5,-1)$

$= \left(\dfrac{40}{27}, -\dfrac{16}{27}, -\dfrac{40}{27}\right)$. $\rightsquigarrow$ $\|\bar{v}_3\| = \dfrac{\sqrt{3456}}{27}$

So an orthonormal basis is

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{27}}, \frac{5}{\sqrt{27}}, -\frac{1}{\sqrt{27}}\right), \left(\frac{40}{\sqrt{3456}}, \frac{-16}{\sqrt{3456}}, \frac{-40}{\sqrt{3456}}\right) \right\}$$

This is an application of:

<u>Souped-Up Projection Theorem</u> Take $\bar{u} \in \mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$. There's a unique way to write $\bar{u} = \bar{w}_1 + \bar{w}_2$ where $\bar{w}_1 \in W$ and $\bar{w}_2 \cdot \bar{w} = 0$ for any $\bar{w} \in W$

$\text{proj}_W \bar{u}$

(Before in Proj. Theorem, $W = \text{span}\{\bar{a}\}$.)

If $\{\bar{v}_1, \dots, \bar{v}_r\}$ is an <u>orthogonal</u> <u>basis</u> for W

then $\text{proj}_W \bar{u} = \text{proj}_{\bar{v}_1} \bar{u} + \text{proj}_{\bar{v}_2} \bar{u} + \ldots + \text{proj}_{\bar{v}_r} \bar{u}$.

(So really Gram-Schmidt says: $\bar{v}_1 = \bar{u}_1$

$\vdots$

$\bar{v}_n = \bar{u}_n - \text{proj}_{\text{span}\{\bar{v}_1, \ldots, \bar{v}_{n-1}\}} \bar{u}$.)

<u>Example</u> Find the projection of $(3,-2,7)$ onto the xy-plane.

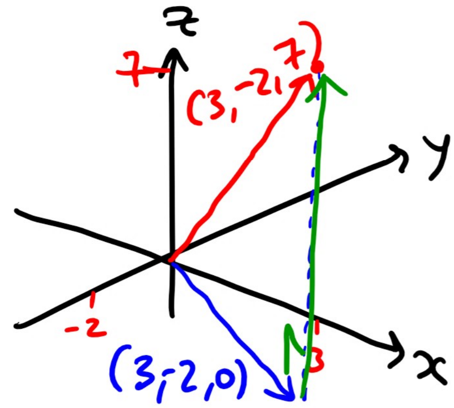
<u>Solution</u> $W = xy$-plane — need an orthogonal basis: $\{\bar{e}_1, \bar{e}_2\}$

$\bar{e}_1 = (1,0,0)$, $\bar{e}_2 = (0,1,0)$

$\text{proj}_W (3,-2,7)$

$= \text{proj}_{\bar{e}_1}(3,-2,7) + \text{proj}_{\bar{e}_2}(3,-2,7)$

$= 3\bar{e}_1 - 2\bar{e}_2 = (3,-2,0)$



<u>Example</u> Find the projection of

$\bar{u} = \begin{bmatrix} 27 \\ 2 \\ -1 \end{bmatrix}$ onto $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0$

<u>Solution</u> ~~$\text{proj}_W \bar{u} = \ldots$~~ NO! It's really tempting to take the basis you're given & start using the formula above right away. But first you need an orthogonal basis.

Find an <u>orthogonal</u> basis for W !!!

Run Gram-Schmidt ... we found last time

$$W = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\bar{v}_1}, \underbrace{\begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}}_{\bar{v}_2} \right\}. \qquad \bar{u} = \begin{bmatrix} 27 \\ 2 \\ -1 \end{bmatrix}$$

Now $\text{proj}_W \bar{u} = \dfrac{\bar{u} \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1 + \dfrac{\bar{u} \cdot \bar{v}_2}{\|\bar{v}_2\|^2} \bar{v}_2$

$$= \frac{\begin{bmatrix} 27 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \frac{\begin{bmatrix} 27 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}}{\underbrace{\left(\frac{21^2 + 12^2 + (-3)^2}{11^2} \right)}_{=11}} \begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 21 \\ 12 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 23 \\ 10 \\ 3 \end{bmatrix}.$$

4.5 Dimension

<u>Def^n</u> V vector space is <u>finite dimensional</u> if it has a finite basis.

Otherwise V is <u>infinite dimensional</u>.

<u>Fact</u> If V has a basis with n vectors

then (1) <u>any</u> set with $> n$ vectors is linearly dependent.

(2) <u>any</u> set with $< n$ vectors does NOT span V.

So if V is finite dimensional, then every basis for V has the same # of elements. This # is <u>dimension of V</u>, called $\dim(V)$.

<u>Notice</u> If you're asked, is $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ a basis for V?

First : count ! Is $r = \dim(V)$?

If no, no hope for S (not a basis for V).

<u>Examples</u> • $\mathbb{R}^n$ has e.g. basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ so $\dim(\mathbb{R}^n) = n$.

• P_d has e.g. basis $\{1, x, x^2, \ldots, x^d\}$ so $\dim(P_d) = d+1$

• $M_{mn}(\mathbb{R})$ has e.g. basis $\left\{ \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix}, \ldots, \begin{bmatrix} \ddots & & \vdots \\ & 0 & 0 \\ \cdots & 0 & 1 \end{bmatrix} \right\}$

so $\dim(M_{mn}(\mathbb{R})) = mn$.

Here's the other Gram-Schmidt Exercise that I promised in class — answer to be posted Monday.

<u>Exercise</u> Find an orthogonal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt Process to $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \\ 2 \end{bmatrix} \right\}$.