1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Exercise from Lecture 32

Exercise Find an orthogonal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt Process to

$$\{(1,0,1),\ (-2,5,-2),\ (4,-10,2)\}.$$

Solution

$$\bar{v}_1 = \bar{u}_1 = (1,0,1).$$

$$\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1}\bar{u}_2 = (-2,5,-2) - \frac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2}\bar{v}_1$$

$$= (-2,5,-2) - \frac{-4}{2}(1,0,1)$$

$$= (-2,5,-2) + 2(1,0,1)$$

$$= (0,5,0).$$

$$\bar{v}_3 = \bar{u}_3 - \text{proj}_{\bar{v}_1}\bar{u}_3 - \text{proj}_{\bar{v}_2}\bar{u}_3$$

$$= (4,-10,2) - \underbrace{\frac{\overbrace{(4,-10,2)\cdot(1,0,1)}^{6}}{2}}_{3}(1,0,1)$$

$$- \underbrace{\frac{\overbrace{(4,-10,2)\cdot(0,5,0)}^{-50}}{25}}_{-2}(0,5,0)$$

$= (4,-10,2) - 3(1,0,1) + 2(0,5,0) = (1,0,-1).$

So an orthogonal basis for $\text{span}\left(\{(1,0,1), (-2,5,-2), (4,-10,2)\}\right)$ is $\{(1,0,1), (0,5,0), (1,0,-1)\}$.

And an orthonormal basis is found using

$$\|(1,0,1)\| = \sqrt{1+1} = \sqrt{2}$$
$$\|(0,5,0)\| = \sqrt{5^2} = 5$$
$$\|(1,0,-1)\| = \sqrt{1+1} = \sqrt{2};$$

we get $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\right\}.$

Method ② gives :

Method ① gives $\bar{v}_1 = \bar{u}_1 = (1,0,1)$.

Method ② gives $\bar{v}_1 = \dfrac{(1,0,1)}{\|(1,0,1)\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$

Now use $\bar{v}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in later steps.

Method ① now gives $\bar{v}_2 = (-2,5,-2) - \dfrac{(-2,5,-2)\cdot\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)}{\left\|\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\right\|^2}\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$

But of course this is 1.

$$= (-2,5,2) - (-\sqrt{2}-\sqrt{2})\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$$
$$= (-2,5,2) + (2,0,2) = (0,5,0).$$

Now method ② gives $\bar{v}_2 = \dfrac{(0,5,0)}{\|(0,5,0)\|} = (0,1,0)$.

We proceed using $\bar{v}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\bar{v}_2 = (0,1,0)$.

$$\bar{v}_3 = (4,-10,2) - \left[(4,-10,2)\cdot\left(\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}\right)\right]\left(\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}\right)$$

divide by 1 →

$$- \left[(4,-10,2)\cdot(0,1,0)\right](0,1,0)$$

divide by 1 →

$$= (4,-10,2) - (2\sqrt{2}+\sqrt{2})\left(\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}\right) + 10(0,1,0)$$

$$= (4,-10,2) - (3,0,3) + (0,10,0)$$

$$= (1,0,-1).$$

Method ② now gives $\bar{v}_3 = \dfrac{(1,0,-1)}{\|(1,0,-1)\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$.

So we end up with the same orthonormal basis, and some parts of the calculations were harder, but some were easier. You do this if you want an orthonormal basis at each step.