1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 33

Last Time Dimension

V - vector space (could be a subspace)

S - $\{\bar{v}_1, \ldots, \bar{v}_n\}$ basis for V

every basis for V has same #

dim(V) = dimension of V = n = # elements in S

If dim$(V) = n$, then
any set with $> n$ elements is linearly DEPENDENT
any set with $< n$ elements does NOT span V

Useful Facts

$\pm$ (Plus - Minus) Theorem S set of vectors in V

+ (a) If S lin. independent & $\bar{v} \notin$ span(S) then $\underbrace{S \cup \{\bar{v}\}}_{\text{add } \bar{v} \text{ to } S}$ is also linearly independent

− (b) If S is lin. dependent & $\bar{v} \in S$ is a linear combination of other vectors in S then

$$\text{span}(S) = \text{span}(\underbrace{S - \{\bar{v}\}}_{\text{delete } \bar{v} \text{ from } S})$$

Hence : Theorem If dim$(V) = n$, and S is a set with n vectors, then

S is a basis $\iff$ S is linearly independent OR S spans V.

inclusive

So to check a set S is a basis for V:

(1) Count: # vectors in S $= \dim(V)$ proceed
$\neq \dim(V)$ — S NOT a basis

(2) Check one of S lin. independent or S spans V.

<u>Example</u> Which of the following are bases for $\mathbb{R}^3$? has $\dim(\mathbb{R}^3) = 3$

$S_1 = \{(1,0,0), (0,1,2)\}$ X S_1 has 2 elements

$S_2 = \{(3,2,1), (-1,0,1), (1,4,7)\}$

$S_3 = \{(1,0,0), (-1,1,0), (0,0,1)\}$

$S_4 = \{(1,0,1), (0,1,0), (1,0,-1), (0,0,1)\}$ X

S_4 has 4 elements

To check S_2, S_3 need to check at least one of "lin. independence" or $\mathrm{span}(S_i) = \mathbb{R}^3$.

We already saw how to check both together: make a matrix with columns the vectors in S_i & check invertibility (i.e. $\det \neq 0$).

This is the matrix way of saying "if # elements in S $= \dim(V)$, so matrix $\square$, then lin. ind. & spanning go together.

S_2 : $\det\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 4 \\ 1 & 1 & 7 \end{bmatrix} = -4 - 10 + 7(2) = 0$

So not invertible

$\Rightarrow S_2$ NOT a basis

S_3 : $\det\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$ so invertible

$\Rightarrow S_3$ IS a basis.

Example Which of the following are bases for P_2?

$\dim(P_2) = 3$

$T_1 = \{x + x^2,\ 1 + x\}$ X ← 2 elements

problem

$T_2 = \{1 - x,\ 1 + x^2,\ x + x^2\}$

$T_3 = \{1,\ x + x^2,\ 1 + x^2\}$

Consider T_2 : $(1-x) - (1+x^2) = -x - x^2$
$= -(x + x^2)$

So T_2 NOT lin. independent so T_2 NOT a basis for P_2.

As for T_3 : check ONE of linear independence OR span $(T_3) = P_2$.

Write : $k_1(1) + k_2(x + x^2) + k_3(1 + x^2) = 0$

& solve for k_1, k_2, k_3 : $k_1 + k_3 = 0 \quad k_2 + k_3 = 0$

(compare coefficients of $1, x, x^2$) $k_2 = 0 \quad \Rightarrow k_1 = k_2 = k_3 = 0.$

So T_3 linearly independent so (since T_3 has 3 elements) T_3 is a basis for P_2.

Other consequences of ± Theorem

If V finite dimensional & S finite set of vectors

then (1) if $\mathrm{span}(S) = V$ but S NOT a basis, we can throw out vectors from S to get a basis for V.

So we can find a basis hiding inside any "too big" set of vectors.

(2) If S is lin. independent but $\mathrm{span}(S) \neq V$ then we can add vectors to S to get a basis for V.

So we can build a basis up out of any linearly independent sets we choose.

And if W is a subspace of V then

(3) $\dim(W) \leq \dim(V)$

(4) $\dim(W) = \dim(V)$ iff $W = V$.

Example Find a basis for

$$V = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$-3\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} =$$

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}$$

not scalar multiples : linearly independent

So $\dim(V) = 2$.

<u>Example</u> Find the dimension of

$$W = \{ a_0 + a_1 x + a_2 x^2 \text{ with } a_0 - 3a_1 = a_2 \}.$$

Exercise : Check W is a subspace of P_2.

① $1+x^2 \in W$ so $W \neq \emptyset$. In fact, $0 \in W$. ✓

② $(a_0 + a_1 x + (a_0 - 3a_1)x^2 + (b_0 + b_1 x + (b_0 - 3b_1)x^2)$
$= (a_0+b_0)x + (a_1+b_1)x + ((a_0+b_0) - 3(a_1+b_1))x^2$ ✓

③ $k(a_0 + a_1 x + (a_0 - 3a_1)x^2)$
$= ka_0 + ka_1 x + (ka_0 - 3ka_1)x^2$ ✓

<u>Solution</u> $W \,?\!=\!?\, P_2$ $\quad 1 + 2x + 3x^2 \notin W$

So $W \neq P_2 \Rightarrow \dim(W) < \dim P_2 = 3$
by (3),(4) above

$$1 + x^2 \in W$$

$$\Rightarrow W \neq \{0\} \Rightarrow \dim W > 0$$
by (3),(4) above

So $\dim W = 1$ or 2

Start with something : $\{1 + x^2\}$. Is this a basis for W?

i.e. is every poly. in W a linear multiple of $1+x^2$?

Rule : $a_0 - 3a_1 = a_2$

Well, $1 + x - 2x^2$ NOT a linear multiple of $1+x^2$

So $1+x-2x^2 \notin \text{span}(\{1+x^2\})$

So by (2) above $\{1+x^2, 1+x-2x^2\}$ is linearly independent

This will be a basis.

span($\{1+x^2, 1+x-2x^2\}$) has $\{1+x^2, 1+x-2x^2\}$ as a basis, so its dim = 2 so dim $W \geq 2$ as this span is a subspace of W; but $\dim(W) \leq 2$ so this means $\dim(W) = 2$ & the only subspace of W of the same dimension is W itself i.e. span($\{1+x^2, 1+x-2x^2\}$) $= W$.

Another method: look at poly. in W:

$$a_0 + a_1 x + (a_0 - 3a_1)x^2$$

$$\rightsquigarrow \underbrace{a_0(1+x^2) + a_1(x-3x^2)}$$

So all polys in W are a linear comb. of $1+x^2$, $x-3x^2$

So now check $\{1+x^2, x-3x^2\}$.

↳ it's linearly independent so $\dim(W) \geq 2$ & so $\dim(W) = 2$

Thus it is a basis (as it is a linearly independent set of vectors in W with same # vectors as $\dim(W)$).

<u>Exercise</u> Find a basis for $\{A \in M_{33}(\mathbb{R})$ with $tr(A) = 0\}$.

(i.e. the set of 3×3 matrices A with $tr(A) = 0$.)