1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

$\binom{\text{WS 19}}{\text{C03}}$ Exercise from Lecture 33

Exercise Find a basis for $\{A \in M_{33}(\mathbb{R})$ with $tr(A) = 0\}$.

(i.e. the set of 3x3 matrices A with $tr(A) = 0$.)

Solution Start by writing the general form of whatever it is that lies in your subspace. Take into account all of the constraints that you need to meet.

Here, that's a 3x3 matrix whose trace is zero.
As you write this out, you'll realize that you have free choice about what goes into all the entries until you get to the very last entry, and then there's a forced move.

You get something like:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -(a+e) \end{bmatrix}.$$

Now try to write this as a linear combination of explicitly given 3x3 matrices. Use the unknown letters of the general form to help you do this. Use them as the coefficients of the linear combination, since they're the numbers over which you have free choice, just like you have free choice over the coefficients when you make a linear combination of basis vectors to get an element of your space.

So here:
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -(a+e) \end{bmatrix} =$$

$$a\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + b\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + c\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + d\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + e\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$
$$+ f\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + g\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + h\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

And now try to fill in what these matrices can be, by comparing what happens in each entry spot on both sides of the equation. You probably get:
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -(a+e) \end{bmatrix} =$$

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$+ f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

So a basis for $\{A \in M_{33}(\mathbb{R})$ with $tr(A)=0\}$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

And the dimension of this subspace of $M_{33}(\mathbb{R})$ is 8 as there are 8 vectors in this, and hence in every, basis.