1ZC3 Engineering Math II-B (Linear Algebra I)

(WS 19 C03) Lecture 34

Yesterday

± Theorem & consequences

V finite-dimensional S finite set of vectors in V

W subspace of V

- If span$(S) = V$ but S too big to be a basis for V, we can throw out any vector that is a linear combination of any others until we get a basis for V
- If S linearly independent but too small to be a basis for V, we can keep adding vectors not in the span until we get a basis for V.
- $\dim(W) \leq \dim(V)$ & $\dim(W) = \dim(V)$ iff $W = V$.

4.7 Row Space, Column Space & Null Space

... of A

Take
$$A = \begin{array}{c} \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \begin{array}{l} \leftarrow \bar{r}_1 \\ \leftarrow \bar{r}_2 \\ \vdots \\ \leftarrow \bar{r}_m \end{array} \\ \underbrace{\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \end{array}}_{\text{column vectors of } A} \end{array}$$

row vectors of A

The row space of A
row(A) = Span$\{\bar{r}_1, \ldots, \bar{r}_m\}$
– subspace of $\mathbb{R}^n$

The column space of A col(A) = span$\{\bar{c}_1, \ldots, \bar{c}_n\}$ — subspace of $\mathbb{R}^m$

Recall null$(A) = \{\bar{x}$ in $\mathbb{R}^n$ with $A\bar{x} = \bar{0}\}$

Recall from 3.4 (Lecture 24) : every solution $\bar{x}$ to $A\bar{x} = \bar{b}$ can be written

$$\bar{x} = \bar{x}_0 + \bar{w}$$

$\bar{w}$ ← some solution to $A\bar{w} = \bar{0}$, i.e. $\bar{w} \in$ null(A)

$\bar{x}_0$ ← particular solution to $A\bar{x} = \bar{b}$

So if $S = \{\bar{v}_1, \dots, \bar{v}_\ell\}$ is a basis for null(A) then, given $\bar{b} \in \mathbb{R}^m$ & particular solution $\bar{x}_0 \in \mathbb{R}^n$ for $A\bar{x} = \bar{b}$, the solution set for $A\bar{x} = \bar{b}$ looks like:

$$\{\bar{x}_0 + k_1\bar{v}_1 + \dots + k_\ell \bar{v}_\ell \ , \text{ for } k_1, \dots, k_\ell \in \mathbb{R}\}$$

How to find a basis for null(A)?

→ it's a basis for the solution space to $A\bar{x} = \bar{0}$.

(We'll come back to this down below.)

Recall We can write $A\bar{x} = \underbrace{x_1\bar{c}_1 + \dots + x_n\bar{c}_n}_{\in \text{col}(A)}$

So $A\bar{x} = \bar{b} \iff 5 \in \text{col}(A)$

[To write $\bar{b} \in \mathbb{R}^m$ as a linear combination of columns of A, solve for $\bar{x}$ in $A\bar{x} = \bar{b}$]

The $x_1, \dots, x_n$ are the coefficients of $\bar{c}_1, \dots, \bar{c}_n$.

Key Facts

If A can be transformed into B by an ERO

↑ Elementary Row Operation

then

(i) $\text{null}(A) = \text{null}(B)$ ← $A\bar{x}=\bar{0} \Leftrightarrow B\bar{x}=\bar{0}$

(we know this one already — row reduction $A \to R$ with a column of zeros at end doesn't change solution space)

(ii) $\text{row}(A) = \text{row}(B)$

(iii) the dependence relations between columns of A are the same as the dependence relations between columns of B

↓ We'll write this more formally at the end.

Example
$$A = \begin{bmatrix} 1 & 4 & 2 & -6 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 14 & 6 & -20 & -4 \\ -1 & -1 & -2 & 3 & 3 \end{bmatrix}.$$

Find bases for $\text{null}(A)$, $\text{row}(A)$ and $\text{col}(A)$ & write the dependence between the columns of A

Solution Take A to RREF:

$$R = \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = s$ $x_4 = t$

To find a basis for $\text{null}(A)$:

solve $A\bar{x}=\bar{0}$: i.e. row reduce $[A \mid 0]$ ↓ $[R \mid 0]$

$x_1 + 2s - 2t = 0 \Rightarrow x_1 = 2t - 2s$

$x_2 - t = 0 \Rightarrow x_2 = t$

$x_5 = 0$

So solution: $\begin{bmatrix} 2t-2s \\ t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

So basis for null(A) = $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To find a basis for row(A):
$\uparrow$
= row(R)

we use the non-zero rows of R:

These are the rows containing the leading 1s.

$\{ [1, 0, 2, -2, 0], [0, 1, 0, -1, 0], [0, 0, 0, 0, 1] \}$

& we can clearly see, looking at those leading 1s coming after 0s, that this set is lin. independent.

To find a basis for col(A):

- we can read off dependence relations between columns of R

$$R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 & \bar{v}_5 \end{matrix} \end{array}$$

Look for leading 1s i.e. pivot columns — they're standard basis vectors

← here $\bar{v}_1, \bar{v}_2, \bar{v}_5$

$\bar{v}_3 = 2\bar{v}_1$

$\bar{v}_4 = -2\bar{v}_1 - \bar{v}_2$

← We can read off a linear expression for each non-pivot column in terms of the pivot columns

Get same dependencies between columns of A:

(same coefficients) $\bar{c}_3 = 2\bar{c}_1 \qquad \bar{c}_4 = -2\bar{c}_1 - \bar{c}_2$

So $\{\bar{v}_1, \bar{v}_2, \bar{v}_5\}$ is a basis for col(R)

So $\{\bar{c}_1, \bar{c}_2, \bar{c}_5\}$ is a basis for col(A)

i.e. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 14 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \\ 3 \end{bmatrix} \right\}.$

Note

col(A) $\neq$ col(R) in general!

<u>Example</u> from last time:

Find a basis for $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

<u>Solution</u> With $A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, col(A) = V

So reduce A to RREF: $\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\uparrow \quad \uparrow$

pivot columns   first 2 columns

So basis for V = col(A)

is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}$ i.e. $\{\bar{c}_1, \bar{c}_2\}$.

$\uparrow \quad \uparrow$

first 2 columns of A

So to summarize: given A to find bases for $\text{null}(A)$, $\text{col}(A)$, $\text{row}(A)$:

① Reduce A to its RREF R

② (a) Solve $A\bar{x} = \bar{0}$ using $[R \mid 0]$;
null(A) basis for sol^n space = basis for $\text{null}(A)$

(b) Non-zero rows of R
row(A) are a basis for $\text{row}(A)$ $(= \text{row}(R))$.

(c) If indices of the pivot columns of R are
col(A) $i_1, \dots, i_\ell$ (e.g. 1st, 3rd, 10th)

then $\{\bar{c}_{i_1}, \dots, \bar{c}_{i_\ell}\}$ ~~are~~ is a basis for $\text{col}(A)$

columns of A

here in this example ($\{\bar{c}_1, \bar{c}_3, \bar{c}_{10}\}$)

& linear relationships between columns of A are true between columns of R (& vice versa).

More formally: if $\bar{v}_r$ is a (non-pivot) column of R with

$$\bar{v}_r = k_{i_1}\bar{v}_{i_1} + \dots + k_{i_\ell}\bar{v}_{i_\ell},$$ where $\bar{v}_{i_j}$ are the pivot columns,

same coefficients ↕ corresponding columns

then $$\bar{c}_r = k_{i_1}\bar{c}_{i_1} + \dots + k_{i_\ell}\bar{c}_{i_\ell}.$$