# 1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 4

**Last time** Matrices are useful!

- For systems of L.E.s, turning the augmented matrix into an RREF matrix tells us about solution(s)
- Gauss-Jordan Elimination : a procedure for turning any matrix into an RREF matrix using Elementary Row Operations

(matrix $\xrightarrow{\text{Gaussian elimination}}$ REF matrix $\rightarrow$ RREF matrix)

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## How many solutions?

With 2 parallel lines e.g. $\begin{cases} x - 2y = -10 \\ 2x - 4y = 6 \end{cases}$

We got

$$\begin{bmatrix} \cancel{1} & -2 & -10 \\ 0 & 0 & \cancel{1} \end{bmatrix} \quad \text{(REF)}$$

$$\text{RREF} \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underbrace{0 \quad 0 \quad 1}_{\text{says } 0 = 1}$

**Point**: if in RREF (or REF) you have a row like $[0 \ldots 0 \; 1]$ – no solutions (inconsistent)

If consistent:

- if there are free variables, then $\infty$-many solutions & RREF gives parameterization
- otherwise one unique solution & you can read it off from the RREF.

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## 1.3 Matrices

An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns

dimensions

**Examples** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a $2 \times 3$ matrix

$\begin{bmatrix} 0 & 5 & -3 & 7 \end{bmatrix}$ is a $1 \times 4$ matrix

sometimes written $(0, 5, -3, 7)$

A $1 \times n$ matrix is called a row matrix or a row vector.

$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a $3 \times 1$ matrix

An $m \times 1$ matrix is a column matrix / column vetor.

A matrix is <u>square</u> if $m = n$ e.g. $\begin{bmatrix} 2 & 5 \\ -\pi & 16 \end{bmatrix}$

is a $2 \times 2$ square matrix

Matrices – capital letters, $A, B, I, \ldots$

Entries – corresponding little letter

$a_{ij}$ = entry in row $i$, column $j$ of matrix $A$

Sometimes we even write $A = [a_{ij}]_{i,j}$.

<u>Example</u> In $A = \begin{bmatrix} 3 & 5 & 2 \\ -4 & 1 & 6 \end{bmatrix}$ $a_{11} = 3$
$a_{12} = 5$
$a_{21} = -4$

For row & column vectors, leave out the unnecessary subscript.

<u>Example</u> $\bar{b} = \underline{b} = \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\cancel{b_{11}}$ $b_1 = 2$
$\cancel{b_{21}}$ $b_2 = 1$
$\cancel{b_{31}}$ $b_3 = 0$.

$\bar{x} = [x_1 \ \ x_2 \ \ x_3]$

<u>Note</u> Two matrices $A, B$ are equal $(A = B)$ when entry for entry, the entries match

i.e. $a_{ij} = b_{ij}$ for every pair $i, j$

(so implicitly A & B must have the same dimensions)

In particular, $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 6 & 4 \end{bmatrix}$ & $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$ even though the systems of L.E.S corresponding to these matrices have the same solutions.

"Is $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1\times 3}$?"

could not possibly be equal.

$A$ $m \times n$ matrix

$A^T$ $n \times m$ matrix : interchange rows & columns of $A$

transpose of $A$

**Example** (i) $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$

(iv) $C = \begin{bmatrix} 1 & 5 & -1 \\ 1 & -8 & 2 \end{bmatrix}$ $C^T = \begin{bmatrix} 1 & 1 \\ 5 & -8 \\ -1 & 2 \end{bmatrix}$

If $A$ is square $\begin{bmatrix} \diagdown \end{bmatrix}$ main diagonal (NW - SE) then the <u>trace</u> of $A$ $\quad tr(A)$ = sum of entries on main diagonal.

<u>Example</u> $\quad A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -1 & 4 \\ -7 & 6 & 5 \end{bmatrix}$

$$tr(A) = 3 - 1 + 5 = 7.$$

$$tr(A^T) = tr\left(\begin{bmatrix} 3 & 2 & -7 \\ 5 & -1 & 6 \\ 7 & 4 & 5 \end{bmatrix}\right) = 3 - 1 + 5 = 7 = tr(A).$$

$\left[\underbrace{\text{If } A \text{ is square}}_{\text{this is another way of communicating "tr(A) is defined"}}\right]$ $\quad tr(A) = tr(A^T)$.

so that I can write down tr(A)!

## Addition and Multiplications with Matrices

<u>Addition</u> – add entries in matching positions

– matrices must have same dimensions

– If $A, B$ are $m \times n$ matrices, then $C = A + B$ is $m \times n$

& $c_{ij} = a_{ij} + b_{ij}$ for all $i, j$

**Example** $A = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & -4 \\ -1 & 2 & 7 \end{bmatrix}$

$$C = \begin{bmatrix} 1+0 & -3+1 & 5-4 \\ -2-1 & 0+2 & 6+7 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 13 \end{bmatrix}$$

## Scalar Multiplication

- multiply all entries in a matrix by <u>same</u> scalar (number)
- $kA$ has entries $ka_{ij}$

**Example** $A = \begin{bmatrix} -2 & 7 \\ 5 & 1 \end{bmatrix}$ $k=3$ ; $3A = \begin{bmatrix} 3(-2) & 3(7) \\ 3(5) & 3(1) \end{bmatrix}$

$$= \begin{bmatrix} -6 & 21 \\ 15 & 3 \end{bmatrix}.$$

## Matrix Multiplication

- Do <u>NOT</u> just multiply entry by entry.

Let $A$ be $m \times n$ matrix

Let $B$ be $\underset{=n}{k} \times l$ matrix.

Product $C = AB$ is ONLY DEFINED

when $n = k$.

# columns of A

# rows of B.

To find $c_{ij}$ take the ith row of A

$$[a_{i1} \; a_{i2} \; \ldots \; a_{in}]$$

& jth ~~row~~ column of B

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

& multiply matching entries and add up:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$