1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 5

Yesterday Matrix Operations

- **Addition** : $C = A + B$ $c_{ij} = a_{ij} + b_{ij}$ A, B, C: same dimensions

- **Scalar Multiplication** : $C = kA$ $c_{ij} = k a_{ij}$

- **Matrix Multiplication** : $C = AB$ A is $m \times n$ (# columns of A); B is $n \times l$ (# rows of B)

$c_{ij} = a_{i1} b_{1j} + \ldots + a_{in} b_{nj}$
- ith row of A: $[a_{i1} \cdots a_{in}]$
- jth column of B: $\begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$

Example $A = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 0 & 4 \end{bmatrix}$

2×2 2×3

2×3

$$C = AB = \begin{bmatrix} -3 & 0 & 12 \\ -3 & 2 & 8 \end{bmatrix}$$

$c_{11} = 0(1) + 3(-1) = -3$ $c_{12} = 0(-2) + 3(0) = 0$

$c_{13} = 0.0 + 3.4 = 12$

$c_{21} = (-1)(1) + 2(-1) = -3$ $c_{22} = (-1)(-2) + 2(0) = 2$

$c_{23} = (-1)(0) + 2(4) = 8$

Example $D = [-3 \quad 2 \quad -1] \qquad E = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$$\underbrace{1 \times 3 \overset{=}{\longleftrightarrow} 3 \times 1}$$

$$DE = [(-3)(2) + 2(-1) + (-1)(0)] = [-8] = -8$$

$$F = ED = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} [-3 \quad 2 \quad -1] = \begin{bmatrix} -6 & 4 & -2 \\ 3 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{3 \times 1 = 1 \times 3}$$

$$f_{11} = 2(-3) \qquad f_{12} = 2(2) = 4$$

Notice DE is dot product of D and E

In general, in $C = AB$,

c_{ij} is the dot product of ith row of A & jth column of B.

Other ways to think about matrix multiplication:

Example

$$\overbrace{\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}}^{B} = \begin{bmatrix} -2 & -5 \\ -1 & 2 \\ 0 & 15 \end{bmatrix}$$

Columns of B $\quad \bar{b}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \bar{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

vectors not entries

$$A\bar{b}_1 = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \leftarrow \text{1st column of } AB$$

$$A\bar{b}_2 = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 15 \end{bmatrix} \leftarrow \text{2nd column of } AB$$

<u>Fact</u> AB defined ; $B = [\bar{b}_1 \ \bar{b}_2 \ \cdots \ \bar{b}_\ell]$

$\uparrow$ column 1 of B

Then AB is the matrix $[A\bar{b}_1 \ A\bar{b}_2 \ \cdots \ A\bar{b}_\ell]$

$\uparrow$ column 1 of AB

Back to example above :

$$A\bar{b}_2 = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 15 \end{bmatrix}$$

$$= 2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$$

A x column vector

is a linear combination (weighted sum) of

the columns of A :

If $A = [A_1 \ A_2 \ \cdots \ A_n]$ & $\bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \leftarrow$ #s not rows

$\uparrow$ column 1 of A ...

then $A\bar{b} = b_1 A_1 + b_2 A_2 + \ldots + b_n A_n$.

1.4 Properties of Matrix Operations

(p.39 longer list)

- $A + B = B + A$ (as long as A, B have same dimensions)
- $(A+B)+C = A+(B+C)$
- $k(A+B) = kA + kB$, scalar k
- $C(A+B) = CA + CB$
- $A(BC) = (AB)C$

Example

$$\overbrace{\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}}^{A}\left(\overbrace{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}}^{B}\overbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}^{C}\right)$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\overbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}^{BC} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$$(AB)C = \left(\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \overbrace{\begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix}}^{AB}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$=$

BUT • Sometimes $AB \neq BA$!!!

We already saw : $ED \neq DE$ (from the Example above)

Example
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

zero matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \neq$$

$A.0 = 0$
(if this makes sense)

This example shows

- Sometimes $AB = 0$ but also $A \neq 0$, $B \neq 0$

We also have :

- Sometimes $AB = AC$ but $B \neq C$

Example
$$\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & 12 \\ -1 & -2 \end{bmatrix}$$

Identities and Inverses

A matrix like
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & 1 & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$$
← square

is an *identity* matrix.

If it is $n \times n$ it is called I_n.
(Sometimes only write I if n obvious from context.)

Example
$$\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$$

$$\left(\underset{\text{Check}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix} = \begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} = \begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix}$$

In general, if A is $m \times n$, then

$$I_m A = A I_n = A$$

So identity matrices are just like 1 in the real #s.