1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 7

Last time ELEMENTARY MATRICES

- Any matrix E you get from any I by applying an Elementary Row Operation

FACT If $I_m \xrightarrow{\text{E.R.O.}} E$ ← elementary

$$\underset{m \times n}{A} \xrightarrow[\text{E.R.O.}]{\text{same}} EA$$

Example $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E$ (elementary)

same ERO

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 5 \\ 6 & -2 & 7 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 0 & 1 \\ 6 & -2 & 7 \\ -1 & 3 & 5 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 5 \\ 6 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 6 & -2 & 7 \\ -1 & 3 & 5 \end{bmatrix}$$

Fact Elementary matrices E are always invertible.

$$I_m \xrightarrow{\text{E.R.O.}} E$$

$$E^{-1} \xrightarrow[]{\text{same E.R.O.}} EE^{-1} = I_m$$

reverse

E.R.O.

To get E^{-1} from I_m do reverse E.R.O. (see last lecture) from E.R.O. used to get E.

In particular E^{-1} is also elementary.

Goal: <u>Algorithm for finding A^{-1}</u>

line is optional

1. If A is $n \times n$, write $\left[\, A \;\middle|\; I_n \,\right]$

2. Take A to RREF using E.R.Os & do same E.R.Os to I_n.

   We get $[\, R \mid B \,]$, where R is RREF of A.

3. Conclude: If $R = I_n$, then $B = A^{-1}$!

   If $R \neq I_n$, then R has a row of zeros, so A is not invertible (i.e. A is singular).

This is not a proper explanation yet! See below for proper explanation.

Why does this work?

Suppose the E.R.Os used to get R from A have elementary matrices called $E_1, E_2, \ldots, E_k$

(k steps ; 1st E.R.O. has el. matrix E_1

2nd ———"——— E_2 ...)

We have:

$$
\begin{array}{l}
\left[\begin{array}{c|c} A & I \end{array}\right] \\
\quad \big\downarrow \text{ 1st E.R.O.} \\
\left[\begin{array}{c|c} E_1 A & \overset{E_1 I}{\overset{\shortparallel}{E_1}} \end{array}\right] \\
\quad \big\downarrow \text{ 2nd ERO} \\
\left[\begin{array}{c|c} E_2 E_1 A & E_2 E_1 \end{array}\right] \\
\quad \big\downarrow \\
\quad \vdots \\
\quad \big\downarrow \text{ kth ERO} \\
\left[\begin{array}{c|c} \overbrace{\underbrace{E_k E_{k-1} \cdots E_2 E_1}_{B} A}^{R} & \overbrace{E_k E_{k-1} \cdots E_2 E_1}^{B} \end{array}\right]
\end{array}
$$

i.e. $R = BA$

So if $R = I$, then $BA = I$ i.e. $B = A^{-1}$!

So $R \overset{\circledast}{=} E_k E_{k-1} \cdots E_2 E_1 A = I$

So if $R = I$, then

And if $R \neq I$, then R not invertible, so A not invertible else we could find R^{-1} using $\circledast$

$$\cancel{E_k^{-1} E_k} E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$

$$\cancel{E_{k-1}^{-1} E_{k-1}} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

i.e. A is product of elementary matrices (& if A is a matrix that happens to be the product of elementary matrices, we can reverse this idea & produce A^{-1} i.e. A must be invertible).

<u>Example</u> $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$. Find A^{-1}!

<u>Solution</u> Write $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$.

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -5 & -2 & 0 & 1 \end{array}\right]$$

El. matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$R_2 \to -R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right]$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to -\frac{1}{3}R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & -1/3 & -1/3 \end{array}\right]$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$E_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow R_2 \to R_2 + 2R_3$

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 4/3 & -5/3 & -2/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & -1/3 \end{array}\right]$$

$$= A^{-1} = E_6 E_5 E_4 E_3 E_2 E_1$$

$$(\text{So } A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}.)$$