1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 8

Yesterday Using Gauss-Jordan Elimination to find INVERSES:

① If A is $n \times n$, write $[A \mid I_n]$.

② Take A to RREF R using Elementary Row Operations $\longrightarrow [R \mid B]$

③ If $R = I_n$, then $B = A^{-1}$!
If $R \neq I_n$, then A not invertible.

We found using this that if $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

then $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 4/3 & -5/3 & -2/3 \\ 2/3 & -1/3 & -1/3 \end{bmatrix}$

Remember: you can check your working!

$$AA^{-1} = A^{-1}A = I$$

1.6 More on Linear Systems & Invertible Matrices

We can write any linear system as a product of matrices:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + \dots + a_{2n}x_n \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} \begin{matrix} = \\ = \\ \vdots \\ = \end{matrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We can write this as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\bar{b}}$$

(m×n)-matrix   n×1   m×1

$$A\bar{x} = \bar{b}$$

Now we see that if A is square $(n \times n)$ we can find a solution if A is invertible:

$$\cancel{A^{-1}A}\bar{x} = A^{-1}\bar{b} \Rightarrow \bar{x} = A^{-1}\bar{b}.$$

Example Solve

$$\begin{aligned} x \qquad + 3z &= 10 \\ -y + 2z &= 2 \\ 2x + y + z &= -3 \end{aligned}$$

Solution in matrix product form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix}$$

We know (from yesterday) that $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 4/3 & -5/3 & -2/3 \\ 2/3 & -1/3 & -1/3 \end{bmatrix}$

So. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -11 \\ 12 \\ 7 \end{bmatrix}.$$

So $x = -11$
$y = 12$
$z = 7$.

The old way:

$$\begin{aligned} x \qquad\quad - 3z &= 10 \\ -y + 2z &= 2 \\ 2x + y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 10 \\ 0 & -1 & 2 & 2 \\ 2 & 1 & 1 & -3 \end{array}\right]$$ ← augmented matrix

& perform EROs to reduce to RREF

→ same EROs used yesterday to find A^{-1}

$\downarrow$ get $\begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$.

The application of these EROs to $\bar{b} = \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix}$ is the same as the matrix product

$$\underbrace{E_6 E_5 E_4 E_3 E_2 E_1}_{A^{-1}} \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix}$$

(For exact values of E_is see yesterday's lecture.)

We saw A invertible means $A\bar{x} = \bar{b}$ has solution $\bar{x} = A^{-1}\bar{b}$.

i.e. at least 1 solution. But:

1 or ∞-many?

<u>Theorem</u> (Gathering facts)

Let A be an $n \times n$ matrix.

The following statements are "equivalent" (all true together, or all false together)

(1) A invertible

(2) The only solution to $A\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.

(3) The RREF of A is I.

(4) A is a product of elementary matrices.

(5) $A\bar{x} = \bar{b}$ is consistent for every choice of $\bar{b}$

(6) $A\bar{x} = \bar{b}$ has exactly one solution for each $\bar{b}$.

<u>Example</u> Solve

$$\begin{aligned} x + y &= 3 \\ x + 2y + 2z &= -1 \\ -x + 2z &= -7 \end{aligned} \quad = \bar{b}_1 \qquad \text{AND} \qquad \begin{aligned} x + y &= 2 \\ x + 2y + 2z &= 7 \\ -x + 2z &= 8 \end{aligned} \quad = \bar{b}_2$$

2 separate systems with same coefficient matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ i.e. $A\bar{x} = \bar{b}_1$ & $A\bar{x} = \bar{b}_2$

Original method (with augmented matrices) but on both systems at the same time:

$$\left[A \mid \bar{b}_1 \mid \bar{b}_2 \right]$$

i.e.

$$\left(\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 1 & 2 & 2 & -1 & 7 \\ -1 & 0 & 2 & -7 & 8 \end{array}\right]\right.$$

2 EROs

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 1 & 2 & -4 & 10 \end{array}\right]$$

1 ERO

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

1 ~~2~~ EROs

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

2nd system inconsistent

2 EROs

$$\left[\begin{array}{ccc|c|c} 1 & 0 & -2 & 7 & 2 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

RREF of A

1st system is consistent; let $z = t$ & then solution is $(7+2t, -4-2t, t)$

What does this tell us about $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$?

Well, since $A\bar{x} = \bar{b}_1$ consistent

& $A\bar{x} = \bar{b}_2$ inconsistent

(5) in Theorem fails, so all (1)-(6) fail, in particular A not invertible.

So for this A, when is $A\bar{x} = \bar{b}$ consistent?

We can write "$A\bar{x} = \bar{b}$"'s augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 1 & 2 & 2 & b_2 \\ -1 & 0 & 2 & b_3 \end{array}\right]$$

← entries of $\bar{b}$

& carry out G-J elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 1 & 2 & b_3 + b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & \boxed{b_3 - b_2 + 2b_1} \end{array}\right]$$

So $A\bar{x} = \bar{b}$ is consistent exactly when $=0$

$$b_3 - b_2 + 2b_1 = 0.$$