1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 9

Last Time

Big Theorem about Invertibility

The following statements are equivalent for any $n \times n$ A:

(1) A is invertible.
(2) The only solution to the system $A\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.
(3) The RREF of A is I_n.
(4) A is a product of elementary matrices.
(5) For all $\bar{b}$, the system $A\bar{x} = \bar{b}$ is consistent.
(6) For all $\bar{b}$, the system $A\bar{x} = \bar{b}$ has exactly 1 solution.

Fact Let A, B $n \times n$. If $AB = I$, then A, B are invertible (with $A^{-1} = B$, $B^{-1} = A$ and $BA = I$).

Why? We'll show B invertible, by showing the only solution to $B\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.

Suppose we have $\bar{x}$ with $B\bar{x} = \bar{0}$.

$$A(B\bar{x}) = A(\bar{0}) = \bar{0}$$

$$I\bar{x} = \bar{0} \Rightarrow \bar{x} = \bar{0}$$

So B is invertible.

Since $AB = I$ & B^{-1} exists $\Rightarrow ABB^{-1} = B^{-1}$.

In a similar way (using (1) & (2) in the Big Theorem) we can show:

<u>Fact</u> If A, B are $n \times n$ and AB is invertible then so are A and B. (<u>Hint</u>: start with B)

<u>Recall</u> To find the inverse of a matrix A we reduced A to its RREF R:

$$[A \mid I] \longrightarrow [R \mid B]$$

If $R = I$, then $B = A^{-1}$.
If $R \neq I$, then A not invertible.

↳ R has row of zeros so R not invertible.

$R = E_k E_{k-1} \cdots E_1 A$

↖ all E_i <u>are</u> invertible

So A not invertible.

<u>Matrix Polynomials</u>

<u>Example</u> Let $A = \begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$

We can for example compute

$$A^2 = \begin{bmatrix} 27 & -8 \\ -16 & 11 \end{bmatrix} \quad \text{or} \quad 8A = \begin{bmatrix} 40 & -8 \\ -16 & 24 \end{bmatrix}$$

& so if $p(x) = x^2 - 8x + 13$ then

it makes sense to talk about $p(A) = A^2 - 8A + 13I$

(for any constant c replace with cI).

$$p(A) = \begin{bmatrix} 27 & -8 \\ -16 & 11 \end{bmatrix} - \begin{bmatrix} 40 & -8 \\ -16 & 24 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We say A is a root of $p(x)$.

$A = \begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$ & $p(x) = x^2 - 8x + 13$.

Notice $\mathrm{tr}(A) = 8$ $\quad$ $\det(A) = 5(3) - (-1)(-2)$
$= 13$

<u>Fun Fact</u> : If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is

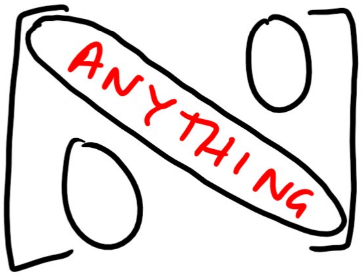
a root of $x^2 - \mathrm{tr}(A)x + \det(A)$

$$x^2 - (a+d)x + (ad - bc)$$

Question for later : How does this relate to inverses?

1.7 Special Kinds of Square Matrices: Diagonal, Triangular, Symmetric

Diagonal $A = [a_{ij}]$ is diagonal if $a_{ij} = 0$ for $i \neq j$



e.g. $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

If A, B are diagonal $n \times n$, then $A \pm B$ and kA are also diagonal.
(k scalar)

If A diagonal $n \times n$ & B is any $n \times n$ matrix

then $AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$

$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}$

(Scale rows of B by corresponding entry of A.)

Likewise BA has columns of B scaled by corresponding entry of A.

e.g.

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ -6 & 1 \end{bmatrix}$$

×3 (row 1), ×1 (row 2)

$$\begin{bmatrix} 2 & 5 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -18 & 1 \end{bmatrix}$$

×3 (column 1), ×1 (column 2)

In particular if A and B are <u>both</u> diagonal

$$AB = \begin{bmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & 0 \\ & 0 & \ddots & \\ & & & a_{nn}b_{nn} \end{bmatrix}$$

← in particular, it's diagonal

Also $$A^k = \begin{bmatrix} a_{11}^k & & & \\ & a_{22}^k & & 0 \\ & 0 & \ddots & \\ & & & a_{nn}^k \end{bmatrix}$$ if A diagonal.

If A diagonal AND invertible, then

$$A^{-1} = \begin{bmatrix} 1/a_{11} & & & 0 \\ & 1/a_{22} & & \\ & 0 & \ddots & \\ & & & 1/a_{nn} \end{bmatrix}$$

If A is diagonal, A is invertible exactly when all $a_{ii} \neq 0$.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ✓ is invertible

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✗ not invertible

<u>Triangular</u> → Upper Triangular $a_{ij} = 0$ if $i > j$
→ Lower Triangular $a_{ij} = 0$ if $i < j$

e.g. $\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$ $\Rightarrow$ U.T.

↖ below diagonal all 0

$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 6 & 0 \\ 1 & -1 & 5 \end{bmatrix}$ → above diagonal all 0 $\Rightarrow$ L.T.

$\begin{bmatrix} 7 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$ → L.T.

$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ → U.T. L.T. Diagonal