# MATH 12<3



$$\begin{array}{c} n\times n \\ \text{matrix} \end{array} \quad I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ 0 & & 1 & & \\ 0 & & & 1 & 0 \\ 0 & 0 & 0 & & 1 \end{bmatrix} \qquad I_n \cdot \underset{n\times k}{A} = A$$

$$A \cdot I_k = A$$

If $A$ is an $n\times n$ matrix, then $A$ is <u>invertible</u> if $A\cdot B = I_n = B\cdot A$ for some $n\times n$ matrix $B$.
If $B$ exists, it is called the inverse of $A$ and we denote it by $A^{-1}$.

e.g. Let $A = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix}$

Claim: $B = A^{-1}$. $AB = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Properties of Inverses.

→ Let $A$ be invertible.

- Then $A^{-1}$ is also invertible and

$$(A^{-1})^{-1} = A$$

- If $k \neq 0$, then $kA$ is invertible and $(kA)^{-1} = \frac{1}{k} A^{-1}$
- $A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_{n \text{ times}}$ is invertible and $(A^n)^{-1} = (A^{-1})^n$

- If $A$ and $B$ are $n \times n$ invertible matrices then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Caution $(AB)^{-1} \neq A^{-1}B^{-1}$ in general

- $A^T$ = transpose of $A$. Then $A^T$ is invertible, if $A$ is, and $(A^T)^{-1} = (A^{-1})^T$

↳ Need to show: $A^T \cdot (A^{-1})^T = I_n$

and $(A^{-1})^T \cdot A^T = I_n$

[Use: $(CD)^T = D^T C^T$]

# 2 Reasons for failing to be invertible

* If an $n \times n$ matrix is <u>not</u> invertible, we say that it is <u>singular</u>

- Reason 1: A has a row or column of zeroes.

eg
$$\underbrace{\begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 5 & 17 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}$$

Must $= 0$

dot product of 2nd row of A and 2 column of B

- Reason 2: A has 2 equal rows or columns.

eg:
$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & -3 \\ 1 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{B} = \begin{bmatrix} 1 & & \\ & & \\ 1 & & \end{bmatrix} \neq I_3$$

dot product of $(1,2,1)$ & 1st column of B

Solving linear systems with inverses.

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned} \iff \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\iff A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

If $A$ is invertible, then

$$\iff A^{-1}\left(A \begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

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ad - bc

The inverse of a 2x2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Fact: $A$ is invertible if and only if $ad - bc \neq 0$

the determinant of $A$

Then $A^{-1} = \left(\dfrac{1}{ad-bc}\right) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Check: $A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc}\right) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & \overset{0}{a(-b)+ba} \\ \underset{0}{cd+d(-c)} & c(-b)+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$c(-b)+d(a)$

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# 1.5 Elementary Matrices and a Method for finding $A^{-1}$

Recall: Elementary Row Operations

① multiply a row by a constant $k \neq 0$

$$R_i \to k \cdot R_i$$

② Add a multiple of one row to another

$$R_i \to R_i + kR_j$$

③ Interchange 2 rows: $R_i \Leftrightarrow R_j$

Every ERO can be reversed:

1. $R_i \to \frac{1}{k} R_i$
2. $R_i \to R_i - kR_j$
3. $R_j \leftrightarrow R_i$

} inverses of ~~the~~ ERO's.

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Elementary Matrix

- An elementary matrix is obtained from $I_n$, by applying a single E.R.O.

eg. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_3 \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

elementary matrices

key Fact: $I_m \xrightarrow{\text{E.R.O}} E \leftarrow$ an elementary matrix

$m \times n$ matrix $A \xrightarrow{\text{same ERO}} EA$

" ~~Element~~ Elementary row operations on A can be accomplished by multiplying A, on the left by the corresponding elementary matrix."

eg: $A = \begin{bmatrix} 1 & \pi & 6 \\ -e & 2 & 7 \\ 0 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & \pi & 6 \\ -e+5 & 2+5\pi & 37 \\ 0 & 8 & 9 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} A \;=$$