Last time : Matrix Representations of Linear Operators

If A and B are similar matrices i.e. there is P with $A = P^{-1}BP$, then A, B represent the same linear operator $T: V \to V$.

Since eigenvalues, determinant, characteristic polynomial are preserved under similarity, it makes sense to define:

$\det(T) = \det([T]_B)$, eigenvalues & eigenvectors of T:

some basis B

$v \in V$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $T(v) = \lambda v$

characteristic polynomial of T: $\det(T - \lambda I) = 0$

SPECIAL MATRICES

We'll bring together: INNER PRODUCT SPACES
- inner products
- orthogonality
- norm, distance, angle

& LINEAR TRANSFORMATIONS
- & matrix representations

Idea : Matrices representing linear transformations on vector spaces with an inner product (inner product spaces) that preserve angle, distance etc.

Recall the matrix Q in a QR decomposition has orthonormal column vectors & satisfies $Q^TQ = I$.

<u>Definition</u> An $n \times n$ matrix A (from now on all matrices are square) is <u>orthogonal</u> if

$$A^{-1} = A^T \qquad (\text{so } A^TA = AA^T = I_n)$$

<u>Theorem</u> A matrix A in $M_n(\mathbb{R})$ is orthogonal exactly when the columns of A form an orthonormal basis for $\mathbb{R}^n$ and the rows of A form an orthonormal basis for $\mathbb{R}^n$.

"Proof": same idea as we had showing $Q^TQ = I$.

<u>Facts</u> If A, B orthogonal, then

(1) A^T is orthogonal (3) AB is orthogonal

(2) A^{-1} is orthogonal (4) $\det(A) = +1$ or -1

<u>Proof</u> (1) If $A^{-1} = A^T$, then $(A^T)^{-1} = A = (A^T)^T$.

(2) If $A^{-1} = A^T$, then $(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$

(or $A^{-1} = A^T$ & we know A^T is orthogonal from (1))

(3) $(AB)^{-1} = B^{-1}A^{-1} = B^TA^T = (AB)^T$

(4) Remember: $\det(MN) = \det(M)\det(N)$ (any M, N)

$$1 = \det(I) = \det(AA^{-1}) = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$$

So $\det(A) = \pm 1$.

Example (1) $A = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ "orthonormal columns & rows" & $\det(A) = -1$

(2) Reflection & Rotation are orthogonal

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (in x-axis)

Suppose $T: V \to V$ is a linear operator, where V is an inner product space, finite-dimensional, where T is represented by an orthogonal matrix.

What effects does T have on V?

First observe:

Theorem If A is an $n \times n$ real matrix, then the following are equivalent:

(1) A orthogonal;

(2) $\|Ax\| = \|x\|$ for each x in $\mathbb{R}^n$;

(3) $Ax \cdot Ay = x \cdot y$ for any x, y in $\mathbb{R}^n$

Proof Observe we can write $u \cdot v$ in terms of "matrices" i.e. $u \cdot v = u^T v$ ← u, v column vectors in $\mathbb{R}^n$

(1) ⇒ (3) A orthogonal. Let x, y be in $\mathbb{R}^n$.

$$Ax \cdot Ay = (Ax)^T Ay = x^T \underbrace{A^T A}_{= I} y = x^T y = x \cdot y.$$

(3) ⇒ (2) $\|Ax\| = \sqrt{Ax \cdot Ax} = \sqrt{x \cdot x}$ (by (3))
$= \|x\|$.

(2) ⇒ (1) Recall Me_j picks out the jth column of M (some matrix)

Also $e_i^T M$ picks out the ith row

So $e_i^T M e_j$ picks out the i,jth entry $(M)_{i,j}$

We want to show $A^T A = I$.

So for each $i = 1, \dots, n$ look at

$$e_i^T (A^T A) e_i = (Ae_i)^T Ae_i = Ae_i \cdot Ae_i = e_i \cdot e_i = 1$$

So the diagonal entries of $A^T A$ are 1.

Now pick $i \neq j$ and note $(e_i + e_j) \cdot (e_i + e_j) = 2$

So

$$2 = (e_i + e_j)^T (e_i + e_j) = (A(e_i + e_j))^T (A(e_i + e_j)) = (e_i + e_j)^T (A^T A)(e_i + e_j)$$
$$= \underbrace{e_i^T (A^T A) e_i}_{=1} + \underbrace{e_j^T (A^T A) e_i + e_i^T (A^T A) e_j}_{\text{Sum to } 0} + \underbrace{e_j^T (A^T A) e_j}_{1}$$

i.e. $(A^T A)_{ji} + (A^T A)_{ij} = 0$

A^TA always symmetric (think about why!)

So $(A^TA)_{ji} = (A^TA)_{ij} = 0$.

So the off-diagonal entries of A^TA are zero. So $A^TA = I$. //

What this shows is that if $T: V \to V$ is a linear operator on a finite dimensional inner products space V represented by an orthogonal matrix, then T preserves lengths (norms) of vectors and hence (by definition) angles & distances between vectors.

<u>Theorem</u> If V is a finite n dimensional inner products space with 2 orthonormal bases B, B', then the transition matrix $[I]_{B',B}$ from B to B' is orthogonal.

To prove this we need:

<u>Facts about orthonormal bases</u>

If $B = \{v_1, \dots, v_n\}$ is an orthonormal basis in an n-dim. real inner product space V, then when we take a vector u in V & look at $[u]_B = (a_1, \dots, a_n)$ in $\mathbb{R}^n$ we have

- $\|u\|_V = \sqrt{a_1^2 + \dots + a_n^2} = \|[u]_B\|$

  norm coming from inner products on V — norm in $\mathbb{R}^n$

- for any 2 vectors u, w in V $d_V(u,w) \overset{\text{Definition}}{=} \|u-w\|_V \overset{\text{By}}{=} \|[u]_B - [w]_B\|$

- and $\langle u, w \rangle_V = [u]_B \cdot [w]_B$.

Proof $\|u\|_V = \sqrt{\langle u, u\rangle_V} = \sqrt{\langle a_1v_1+\ldots+a_nv_n, a_1v_1+\ldots+a_nv_n\rangle_V}$

$$= \sqrt{a_1^2\underbrace{\cancel{\langle v_1, v_1\rangle}}_{=1} + \cancel{a_1a_2\langle v_1, v_2\rangle} + \ldots\ldots + a_n^2\underbrace{\cancel{\langle v_n, v_n\rangle}}_{=1}}$$

$$= \sqrt{a_1^2+\ldots+a_n^2}.$$

Rest left as an exerise.

Back to the theorem:

Proof Want to show $[I]_{B',B}$ is orthogonal.

We'll use the Theorem above & show

$$\|[I]_{B',B}x\| = \|x\| \text{ for all } x \text{ in } \mathbb{R}^n.$$

Let x be in $\mathbb{R}^n$ and then take w to be the vector in V with coordinates x i.e. $[w]_B = x$

Then $\|w\|_V = \|[w]_{B'}\|$ by Fact above as B' is orthonomal

$$= \|[I]_{B',B}[w]_B\| = \|[I]_{B',B}x\|$$

But also $\|w\|_V = \|[w]_B\|$ by Fact above as B is orthonomal

$$= \|x\|.$$

So $I_{B',B}$ is orthogonal. //