MATH 2R03 (Linear Algebra II) -- 31 July 2018 -- Lecture 13 (Review)

Suggested Topics

§8 – Linear Transformations, Rank, Nullity, (Kernel & Range), 1-1, onto, proofs ...

§§8.4 – 8.5 – Matrices for Linear Transformations (how to find them & work with them)

§§7.2, 7.5 – Orthogonal/Unitary Diagonalization

§§6.4, 6.6 – Projection as Best Approximation

§§7.3, 7.4 – Quadratic Forms

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§8 Linear Transformations

$$T : U \longrightarrow V, \quad U, V \text{ vector spaces}$$

satisfying ① $T(u_1+u_2) = T(u_1)+T(u_2)$

② $T(ku) = kT(u)$

for all u_1, u_2, u in U and scalars k

From ① & ② immediately get $\underset{\text{in } U}{T(0)} = \underset{\text{in } V}{0}$

(so ~~maybe~~ checking $T(0) =$ something else tells you T not linear)

& also we get from ① & ② $T(-u) = -T(u)$ and $T(u_1 - u_2) = T(u_1) - T(u_2)$

T has a **kernel**: $\ker(T)$, all the vectors in U that get sent by T to 0 i.e. all u in U with $T(u) = 0$.

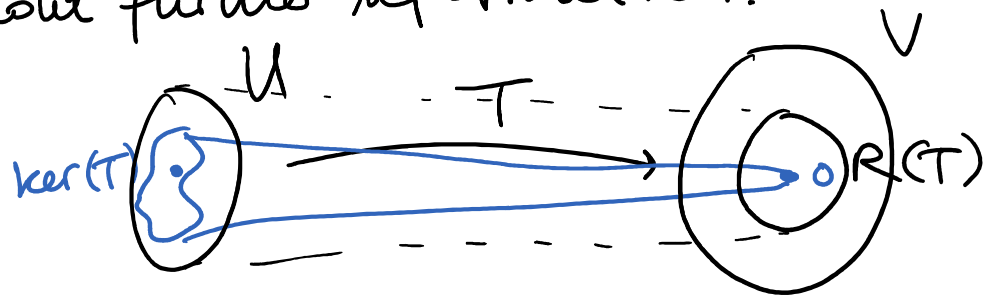
T has a **range**: $R(T)$, all the vectors in V which get hit by T i.e. all v in V for which there is some u in U for which $T(u) = v$.

Now suppose we have $T: U \to V$ with U and V finite dimensional.

We have **Rank – Nullity Theorem**:

$$\underset{\text{domain}}{\dim(U)} = \underbrace{\dim(\ker(T))}_{\text{nullity}(T)} + \underbrace{\dim(R(T))}_{\text{rank}(T)}$$

Does not tell you what dim V is exactly without further information.



nullity(T) = dim (ker(T)) is a measure of how much T "collapses" U into $R(T)$ (inside V).

If nullity$(T) = 0$, then $\dim(U) = \dim(R(T))$
(U does not change dimension after T is applied)

This happens exactly when T is 1-1.
(This is when we have $\ker(T) = \{0\}$.)

If nullity$(T) > 0$ we have some "collapse"
(as $\dim(U) > \dim R(T)$
i.e. T NOT 1-1

Notice that so far everything is true whether T is onto or NOT.
($V = R(T)$) ($V \neq R(T)$)

So what if T is onto? $\leftarrow V = R(T)$

If $\text{nullity}(T) > 0$ i.e. T is NOT 1-1

then
$$\begin{aligned} \dim(U) &= \text{nullity}(T) + \dim R(T) \\ &= \text{nullity}(T) + \dim(V) \\ &> \dim(V) \end{aligned}$$

In particular T is onto but $\dim(U) \neq \dim(V)$.

e.g. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$$(x,y,z) \longrightarrow (x,y)$$

And if T is 1-1 ($\text{nullity}(T) = 0$)

but T is NOT onto ($R(T) \neq V$)

[$R(T)$ a subspace of V so $\dim R(T) < \dim(V)$]

So Rank-Nullity Theorem tells us
$$\begin{aligned} \dim(U) &= \text{nullity}(T) + \text{rank}(T) \\ &= 0 + \dim(R(T)) \\ &< \dim(V) \end{aligned}$$

So again, although T is 1-1, T not being onto means in particular $\dim(U) \neq \dim(V)$.

e.g. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^5$ or

$(x,y,z) \longmapsto (x,y,z,0,0)$

e.g. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ Put the (x,y)-plane (2D object) inside 3-space $\mathbb{R}^3$

$(x,y) \longmapsto (x,y,0)$

Note: $\dim \ker(T) = \text{nullity}(T)$ can be any of $0, 1, 2, \ldots, \dim(U)$

$\dim R(T) = \text{rank}(T)$ can be any of $0, 1, 2, \ldots, \dim(V)$

(So Q: If T is 1-1 & not onto is $\dim(U) < \dim(V)$.)

ANS: Yes, see above

Note also $\dim(U) = \text{nullity}(T) + \text{rank}(T)$

$= \dim \ker(T) + \dim R(T)$

$\geq \dim R(T)$

So in fact ~~$\dim(U)$~~ $\dim R(T)$ can be any of $0, \ldots,$ ~~$\dim R(T)$~~ $\dim U$.

The Take-Home is this:

the only situation where T 1-1 $\Rightarrow$ T onto or T onto $\Rightarrow$ T 1-1

is when you are already given $\dim(U) = \dim(V)$!

But if T 1-1 AND onto then $\dim(U) = \dim(V)$

(By Rank-Nullity Theorem.)

Proofs involving linear transformations

e.g. we had that if $T: U \to V$ is 1-1 then we can define an inverse $T^{-1}: R(T) \to U$ and it is linear & 1-1.

How to show these?

For Linear, check each of ① and ② i.e.

want to show:

① for each v_1, v_2 in $R(T)$, $T^{-1}(v_1+v_2)$
$$= T^{-1}(v_1) + T^{-1}(v_2)$$

& ② for each v in $R(T)$ and each scalar k,
$$T^{-1}(kv) = k\,T^{-1}(v).$$

(Write down what you want to show as precisely as possible.)

For ①, you might try to say "$v_1 + v_2$ is in $R(T)$, so there is u in U with $T(u) = v_1 + v_2$. So $u = T^{-1}(v_1+v_2)$ ..."

but how does this relate to $T^{-1}(v_1)$ or $T^{-1}(v_2)$?

Not clear. Let's try the right-hand side of our equation:

v_1 and v_2 are in $R(T)$. So there are u_1, u_2 in U with $T(u_1) = v_1$ and $T(u_2) = v_2$.

Then $T(u_1) + T(u_2) = v_1 + v_2$

and $T(u_1 + u_2) = v_1 + v_2$ by linearity of T.

So $T^{-1}(v_1+v_2) = u_1+u_2 = T^{-1}(v_1) + T^{-1}(v_2)$. //

(2) - exercise, similar to (1).

T^{-1} 1-1 : To show some map $S: X \to Y$ is 1-1 a good strategy is to take 2 vectors in domain (x_1, x_2 say) and suppose $S(x_1) = S(x_2)$ (same image) and show $x_1 = x_2$ (so S must be 1-1).

So for $T^{-1} : R(T) \to U$: take $v_1, v_2 \in R(T)$ and suppose $T^{-1}(v_1) = T^{-1}(v_2)$

(want to show $v_1 = v_2$.)

v_1, v_2 in $R(T)$ means there are u_1, u_2 in U with $T(u_1) = v_1$ and $T(u_2) = v_2$.

Since therefore $T^{-1}(v_1) = u_1$ and $T^{-1}(v_2) = u_2$

we have $u_1 = u_2$

so $\underset{v_1}{\underset{\parallel}{T(u_1)}} = \underset{v_2}{\underset{\parallel}{T(u_2)}}$. //

Matrices for Linear Transformations

<u>SETUP</u> $T: U \longrightarrow V$ (maybe $U = V$, maybe not)

Given bases B_U for U & B_V for V (if $U = V$ then maybe $B_U = B_V$ but maybe not)

<u>Question is:</u> find $[T]_{B_V, B_U}$ and ...

e.g. compute $T(\ldots)$ or find $\det(T)$ or eigenvalues for T or ...

Key 1st step: find $[T]_{B_V, B_U}$.

<u>Main idea:</u> Write $T(u_i)$ for each basis vector u_i in B_U in terms of the basis B_V.

<u>Examples:</u> Ch. 8.5 Q9

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$B' = \{\underbrace{(-2,1,0)}_{v_1}, \underbrace{(-1,0,1)}_{v_2}, \underbrace{(0,1,0)}_{v_3}\}$

B standard basis i.e. $\{(1,0,0), (0,1,0), (0,0,1)\}$

$T(x_1, x_2, x_3) = (-2x_1 - x_2, x_1 + x_3, x_2)$

To find $[T]_{B',B}$,

<u>Step 1</u> Plug in each vector from B into T &

<u>Step 2</u> write it as a linear combination of the vectors in B'

$$T(1,0,0) = (-2, 1, 0) = 1 \cdot v_1 + 0 v_2 + 0 v_3$$
$$T(0,1,0) = (-1, 0, 1) = 0 v_1 + 1 \cdot v_2 + 0 v_3$$
$$T(0,0,1) = (0, 1, 0) = 0 v_1 + 0 v_2 + 1 \cdot v_3$$

Then $[T]_{B',B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(So <u>Step 3</u>: write coefficients of representation from Step 2 as <u>columns</u> of matrix.)

<u>Example §8.5 Q21</u>

$T: P_2 \longrightarrow P_2 \qquad T(p(x)) = p(x-1).$

Q says "Find det(T) & eigenvalues of T."

First step: find a matrix for T with respect to <u>some</u> bases for P_2.

I'm going to use basis $B = \{1, x, x^2\}$ for P_2 on both sides.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; 5 \quad\; x \quad x^2$

Step 1 $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ Step 2

$$T(\overset{5}{1}) = 1^5 = 1$$
$$T(x) = x-1 = -1 + x$$
$$T(x^2) = (x-1)^2 = 1 - 2x + x^2$$

Step 3 $\quad [T]_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

$T(p(x)) = p(x-1)$

Let's suppose we want to use this to find $T(\underbrace{5+9x-7x^2}_{\text{has coordinates wrt } B})$:

Solution

$$[5 \quad 9 \quad -7]$$

So we find $[T(5+9x-7x^2)]_B$

$$= [T]_B \, [5 \quad 9 \quad -7]^T$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 9 \\ -7 \end{pmatrix} = \begin{pmatrix} -11 \\ 23 \\ -7 \end{pmatrix}$$

Now convert this back to a polynomial & we get $T(5+9x-7x^2) = -11 + 23x - 7x^2$.

Orthogonal / Unitary Diagonalization

Orthogonal: $A^{-1} = A^T$ (over $\mathbb{R}$)

Unitary: $A^{-1} = \overline{A}^T = A^*$ (over $\mathbb{C}$)

In both cases rows form an orthonormal set & columns form an orthonormal set.

i.e. if $A = \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix}$ then $c_i \cdot c_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$

& same, if $A = \begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$ then $r_i \cdot r_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$

Example §7.5 Q 14.

Find a unitary matrix P which diagonalizes A and determine $P^{-1}AP$ where

$$A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}.$$

Solution

Strategy: **Step 1** Find eigenvalues.

Step 2 Find a basis for each eigenspace.

Q: Are there n (here 2) linearly independent eigenvectors?

As long as A is symmetric ($A = A^T$) over $\mathbb{R}$ or A is Hermitian ($A = \bar{A}^T (= A^*)$) or normal ($AA^* = A^*A$) over $\mathbb{C}$ then this answer <u>will be</u> yes (& therefore A <u>is</u> diagonalizable & can proceed).

(In this case A is Hermitian so we'll be OK.)

<u>Step 3</u> Run Gram-Schmidt Process on each basis for an eigenspace that we found in Step 2 to get an orthonormal set of eigenvectors.

<u>Step 4</u> Form P with columns these eigenvectors; and $P^{-1}AP$ is the diagonal matrix with entries eigenvalues from Step 1 inserted in same order as corresponding columns of P.

So for our $A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}$,

<u>Step 1</u> Find eigenvalues:

Solve characteristic equation

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -i \\ i & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)^2 - (-i)(i)$$

$$= 9 - 6\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4).$$

So $\lambda = 2, 4$.

<u>Step 2</u> Find eigenvectors. Solve $(A - \lambda I)x = 0$ for each λ.

<u>$\lambda = 2$</u>: Solve $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} x = 0$

Gaussian elimination $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$

(second row is $i \times$ first row)

tells us $x_1 = i x_2$

So basis is e.g. $\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$

$\underline{\lambda = 4}$: Solve $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} x = 0$

Gaussian elimination e.g. $\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

2nd row is $i \times$ (1st row)

So we get $x_1 = -ix_2$

& basis is e.g. $\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$.

<u>Step 3</u> Normalize vectors in Step 2 (when there's only 1 vector in each basis that's the only step of Gram-Schmidt Process that you need to do).

We get orthonormal eigenvectors

$$\left\{ \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$$ and so

<u>Step 4</u> $P = \begin{pmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ and

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

If I'd taken $P = \begin{pmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ then $P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$.

Normalizing vectors in $\mathbb{C}^2$ say

e.g. $\begin{pmatrix} i \\ 1 \end{pmatrix}$ This has norm $\sqrt{\begin{pmatrix} i \\ 1 \end{pmatrix} \cdot \begin{pmatrix} i \\ 1 \end{pmatrix}}$

$$= \sqrt{i(-i) + 1 \cdot 1}$$

$$= \ldots$$

don't forget $\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow$ has norm $\sqrt{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}$

$$= \sqrt{x\bar{x} + y\bar{y}}$$

$$= \sqrt{|x|^2 + |y|^2}.$$

Projection as Best Approximation

eg Least squares Problem

System $Ax = b$ possibly not soluble

Want the vector x which comes "closest" to being a solution i.e. minimises

$$\|Ax - b\|.$$

We get this "approximating" x by

taking $Ax-b$ in $W^\perp$, where (W is col(A)

so $W^\perp = \text{null}(A^T)$

So in particular we want

$$A^T(Ax-b) = 0$$

i.e. $A^TAx = A^Tb$.

For functions given $W = \text{Span}\{\text{some functions}\}$

& want to approximate $f(x)$ as close as possible by an element of W.

The solution is given by $\text{proj}_W f$.

e.g. $f(x) = x^2$, $W = \text{span}\{e^x, e^{2x}\}$

To find the best approximation to $f(x) = x^2$ lying in W, find $\text{proj}_W x^2$

— Find $\text{proj}_W x^2 = \dfrac{\langle x^2, v_1\rangle}{\|v_1\|^2} v_1 + \dfrac{\langle x^2, v_2\rangle}{\|v_2\|^2} v_2 + \ldots$

where $\{v_1, v_2, \ldots\}$ is an orthogonal basis for W.

If $\langle\ ,\ \rangle$ is integral inner product on $[0,1]$

then $\langle e^x, e^{2x} \rangle = \int_0^1 e^{3x}\,dx = \left[\frac{1}{3}e^{3x}\right]_0^1 \neq 0$

So $\{e^x, e^{2x}\}$ not orthogonal basis for W

So run Gram-Schmidt to get one.