Review of complex #s

Problem: Not all equations involving real numbers have real solutions e.g. $3x^2 - x + 2$

Quadratic formula: $x = \dfrac{1 \pm \sqrt{1-4.3.2}}{6}$

$= \dfrac{1 \pm \sqrt{-23}}{6}$

(There is no real # which, when squared, gives -23.)

Introduce a new number system by introducing one new number i which satisfies $i^2 = -1$

— this new system has solutions to all the polynomials with coefficients in this system.

The result: the complex numbers $\mathbb{C}$

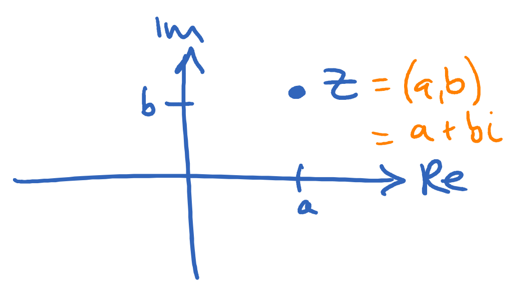
Numbers of the form $z = \underbrace{a}_{\text{Real component / Real part of } z} + \underbrace{b}_{\text{Imaginary part of } z}i$ where a, b are real numbers

So our quadratic above has solutions

$$\frac{-1 \pm \sqrt{-23}}{6} = -\frac{1}{6} \pm \frac{\sqrt{23}}{6} i$$

Alternative notation $z = (a, b) \in \mathbb{R}^2$

Think of $\mathbb{C}$ as a plane:



But we'll use vectors a lot for other things, so we'll stick with the $a+bi$ notation.

Operations using complex numbers

Addition

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$ (gather components)

Multiplication

$$(a+bi)\cdot(c+di) = ac+(bc+ad)i+\underbrace{bdi^2}_{-bd}$$

$$= (ac-bd)+(bc+ad)i$$

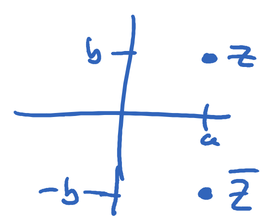
(Do exactly what you expect & multiply out brackets, remembering $i^2=-1$.)

Division Take two complex #s $z_1 = a+bi$, $z_2 = c+di$

If $z_2 \neq 0$, what is $\frac{z_1}{z_2}$?

Useful notation: conjugate of a complex #

$z = a+bi$ is $\overline{z} = a-bi$

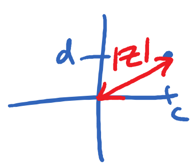


Want to find $\frac{z_1}{z_2} = \frac{a+bi}{c+di}$

Multiply top and bottom by $\overline{z_2}$: $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{\underbrace{z_2\overline{z_2}}}$ ← 2 complex #s multiplied together

$z_2\overline{z_2} = (c+di)(c-di)$ → real number

$$= \underbrace{c^2+d^2}_{|z_2|^2}$$



$$= \frac{(a+bi)(c-di)}{c^2+d^2}$$

In particular: $z_2^{-1} = \frac{1}{z_2} = \frac{c-di}{c^2+d^2} = \frac{\overline{z_2}}{|z_2|^2}$

Exercises

Let $z_1 = 6 + i$, $z_2 = -2 - 3i$, $z_3 = 1 - 2i$

Find (a) $z_1 - z_3$ (b) $\overline{\overline{z_3}}$ (c) $z_1 \cdot \overline{z_2}$

(d) $\overline{z_1} \cdot z_2$ (e) $\dfrac{z_2}{z_3}$.

Sample answers:

(b) $\overline{\overline{z_3}} = \overline{\overline{(1-2i)}} = \overline{(1+2i)} = 1 - 2i$.

(a) $5 + 3i$ (b) $1 - 2i$ (c) $-15 + 16i$ (d) $-15 - 16i$

$$\overline{z_1 \cdot \overline{z_2}} = \overline{z_1} \cdot z_2$$

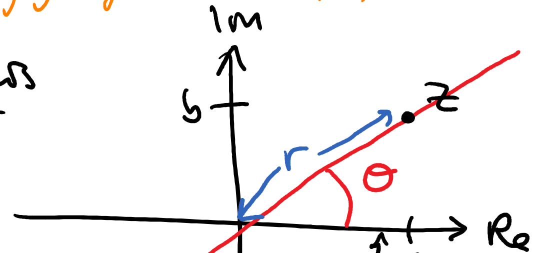
(e) $\frac{4}{5} - \frac{7}{5} i$

(uses $z_3^{-1} = 1/5 + 2/5\, i$)

(Remember, in general: $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$)

(So the solution to (d) is just the conjugate of the solution to (c).)

Polar form of complex numbers

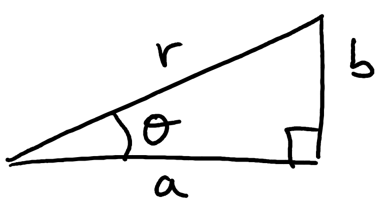


Can represent $z = a + bi$

in terms of argument θ

(angle between real line & positive Real axis)

and modulus r $(= |z|)$



Using this triangle we can find

$r^2 = a^2 + b^2$ so $r = \sqrt{a^2 + b^2}$

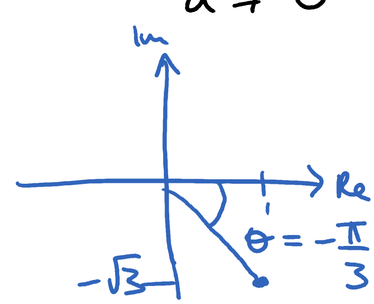
and $\theta = \arctan\left(\frac{b}{a}\right)$ (where this makes sense $a \neq 0$)

e.g. $z = 1 - \sqrt{3}\, i$ has

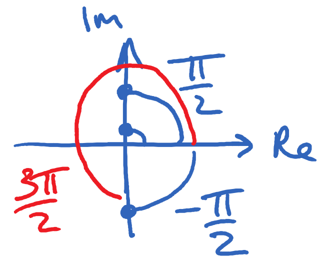
$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

see next page $\longrightarrow \left(-\frac{\pi}{3} + 2\pi\right)$



If $a = 0$,
$z = bi$
So $\theta =$

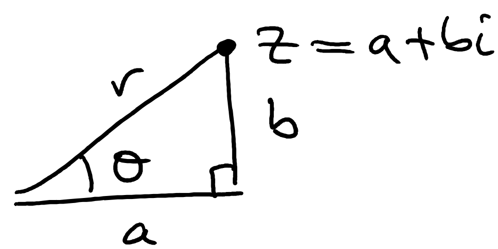


b positive, $\pi/2, \ldots$
b negative, $-\pi/2, \; 3\pi/2 \; (= -\pi/2 + 2\pi), \ldots$

In above case, this ← case & all cases: Add or subtract any multiple of 2π and gets another argument for the same point.

<u>Principal argument</u> for z: the argument for z lying in $(-\pi, \pi]$

Looking at the triangle again



$a = r\cos\theta$, $b = r\sin\theta$

$$\text{So} \quad z = r\cos\theta + i r\sin\theta$$
$$= r\underbrace{(\cos\theta + i\sin\theta)}.$$

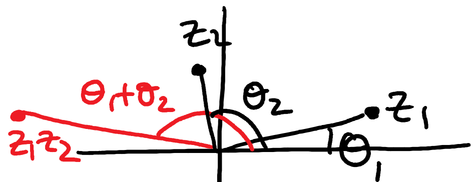
Using Maclaurin series we can show that this equals $e^{i\theta}$ (Euler's formula)

So $z = re^{i\theta}$ where $r = |z|$, $\theta = \arg(z)$

Multiplication & Division revisited

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

Multiply: $z_1 \cdot z_2 = \underbrace{r_1 r_2}_{\text{multiply moduli}} e^{i(\overbrace{\theta_1+\theta_2}^{\text{add arguments}})}$ $= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$



$|z_1 z_2| = |z_1||z_2|$

Division: $\dfrac{z_1}{z_2} = \underbrace{\dfrac{r_1}{r_2}}_{\text{divide moduli}} e^{i(\overbrace{\theta_1-\theta_2}^{\text{subtract arguments}})}$ $= \dfrac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

In general $z^n = r^n e^{in\theta} = r^n \underline{(\cos(n\theta) + i\sin(n\theta))}$

$(z = re^{i\theta})^n$

$= (r(\cos\theta + i\sin\theta))^n = r^n \underline{(\cos\theta + i\sin\theta)^n}$ // The equality of the two expressions underlined in red is called **De Moivre's Formula**

for integers n.

What about roots $z_1^{1/n}$? (n non-zero integer)

We want solutions z_2 to $z_2^n = z_1$

i.e. r_2, θ_2 such that $\underbrace{r_2^n e^{in\theta_2}}_{z_2^n} = z_1 = r_1 e^{i\theta_1}$

So $r_2^n = r_1 \rightsquigarrow r_2 = \sqrt[n]{r_1}$. (Compare moduli.)

We need θ_2 to satisfy $n\theta_2 = \theta_1$ (Compare arguments.)

But infinitely many possibilities for θ_1

Call the principal argument of z_1 α_1

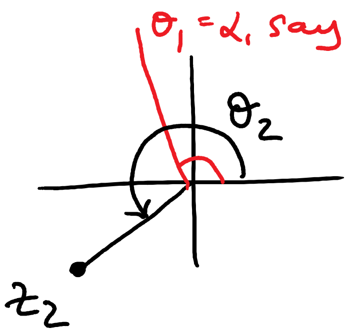
Then θ_1 could be $\alpha_1 + 2\pi k$ for any integer k.

So $\theta_2 = \frac{\theta_1}{n} = \frac{\alpha_1 + 2\pi k}{n} = \frac{\alpha_1}{n} + \frac{2\pi k}{n}$ ~~for any integer k.~~

actually $k = 0, \dots, n-1$ enough to get all solutions

$\underset{k=0}{\frac{\alpha_1}{n}}, \underset{k=1}{\frac{\alpha_1}{n} + \frac{2\pi}{n}}, \dots, \underset{k=n-1}{\frac{\alpha_1}{n} + \frac{2\pi(n-1)}{n}}$ are all different numbers, but $\frac{\alpha_1}{n} + \frac{2\pi \cancel{n}}{\cancel{n}}$ is same as $\frac{\alpha_1}{n}$ so we start the list over again at $k = n$

So $z_2 = \sqrt[n]{r_1}\left(\cos\left(\frac{\alpha_1}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\alpha_1}{n} + \frac{2\pi k}{n}\right)\right)$



z_2 has arg. θ_2

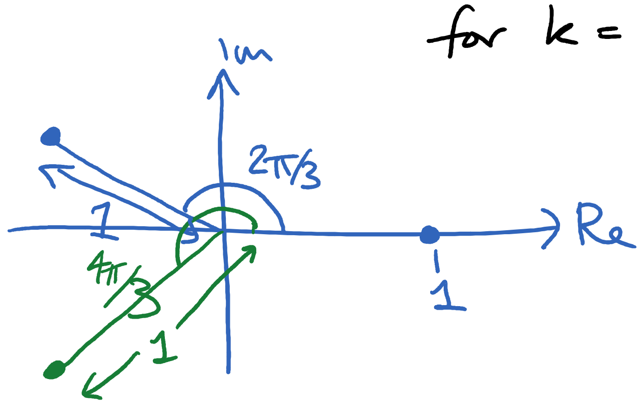
z_2^n has arg. $n\theta_2$ which could have principal argument smaller than θ_2

Special case "Roots of unity" i.e. z such that $z^n = 1$. (for some non-zero integer n)

↑ nth root of unity

Since $1 = e^0$, the nth roots of unity are $e^{\frac{2\pi k}{n} i}$ for $k = 0, \dots, n-1$.

e.g. $n = 3$



Exercises (a) Find the square roots of $-\sqrt{3} + i$

(b) Find the fifth roots of $6i$.