Exercises from the end of Lecture 1 on complex roots

(a) Find the square roots of $-\sqrt{3}+i$.

Solution

First we need to put $-\sqrt{3}+i$ into polar form:

modulus $r = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2.$

argument $\theta = \arctan\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$ ← We take the argument in $[0, 2\pi)$.

The goal is to find z such that
$$z^2 = -\sqrt{3}+i = 2\left(\cos\left(\tfrac{5\pi}{6}\right) + i\sin\left(\tfrac{5\pi}{6}\right)\right).$$

Using the formula we derived in class, we have that

$$|z| = \sqrt{2} \quad \text{and} \quad \arg(z) = \frac{\frac{5\pi}{6} + 2\pi k}{\underbrace{2}_{n=2\text{ here}}}, \text{ for } k = 0, \underbrace{1}_{n-1=1\text{ here.}}$$

$$= \frac{5\pi}{12} + \pi k, \text{ for } k = 0, 1$$

$$= \frac{5\pi}{12}, \frac{17\pi}{12}.$$

So, packaging this all together, we get that the square roots of $-\sqrt{3}+i$ are: $z = \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$

and $z = \sqrt{2}\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right)$.

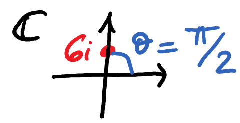
(b) Find the fifth roots of $6i$.

<u>Solution</u> We follow the same procedure as in (a).

First put $6i$ in polar form:

modulus: $\sqrt{6^2} = 6$.

argument: here we cannot use our usual formula involving arctan, since we cannot divide by $a=0$. But draw a picture:



and we see that an argument of $6i$ is $\frac{\pi}{2}$.

We want to find values of z with $z^5 = 6i = 6\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$.

Using the formula from class, we have that, for such z,

$|z| = \sqrt[5]{6}$ and z has argument $\dfrac{\frac{\pi}{2} + 2\pi k}{5}$, $\leftarrow n = 5$ here.

for $k = 0, 1, 2, 3, 4$.

i.e. argument $\frac{\pi}{10}$, $\frac{\pi}{10} + \frac{2\pi}{5} = \frac{\pi}{2}$, $\frac{\pi}{10} + \frac{4\pi}{5} = \frac{9\pi}{10}$,

$\frac{\pi}{10} + \frac{6\pi}{5} = \frac{13\pi}{10}$, $\frac{\pi}{10} + \frac{8\pi}{5} = \frac{17\pi}{10}$.

Thus the fifth roots of $6i$ are $z = \sqrt[5]{6}\left(\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right)$,

$z = \sqrt[5]{6}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$, $z = \sqrt[5]{6}\left(\cos\left(\frac{9\pi}{10}\right) + i\sin\left(\frac{9\pi}{10}\right)\right)$,

$z = \sqrt[5]{6}\left(\cos\left(\frac{13\pi}{10}\right) + i\sin\left(\frac{13\pi}{10}\right)\right)$, $z = \sqrt[5]{6}\left(\cos\left(\frac{17\pi}{10}\right) + i\sin\left(\frac{17\pi}{10}\right)\right)$.