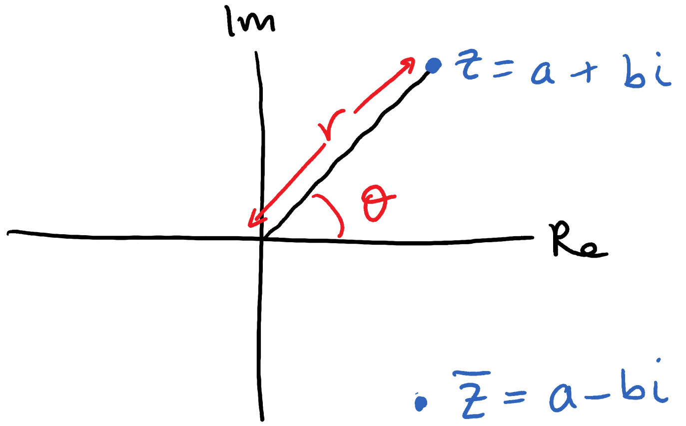
# MATH 2R03 (Linear Algebra II) -- 21 June 2018 -- Lecture 2

**Last time:** Review of complex numbers! $\mathbb{C}$



Modulus:

$$r = \sqrt{a^2 + b^2} = |z|$$
$$= \sqrt{z\bar{z}}$$

Argument:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

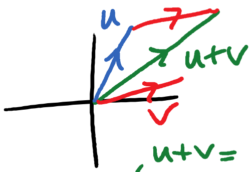
## Review of vector spaces

- collections of objects that interact with one another the same way vectors do (in the following sense):

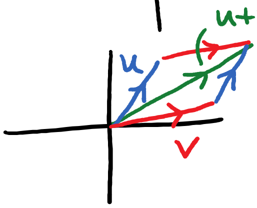
$$\left.\begin{array}{l} V - \text{objects} \\ \_ + \_ \\ k\cdot\_ \text{ for each scalar } k \end{array}\right\} \text{These are specified/named/defined.}$$

$V$ is a vector space (with this addition $+$ and scalar multiplication $k\cdot$ for each $k$) if the following axioms are all true:

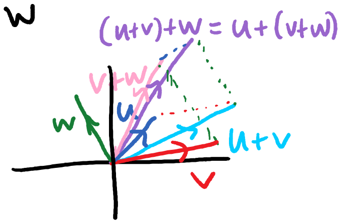
1. If $u, v$ in $V$, then $u+v$ is in $V$



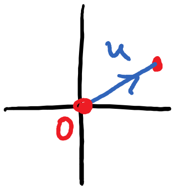
2. If $u, v$ in $V$, then $u+v = v+u$



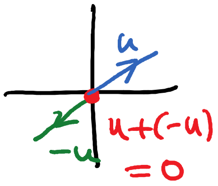
3. If $u, v, w$ in $V$, then $u+(v+w) = (u+v)+w$



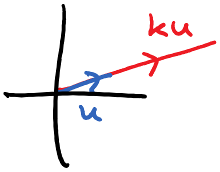
4. There is an object in $V$, called $0$, with the property that $0+u = u$ (for any $u$ in $V$)



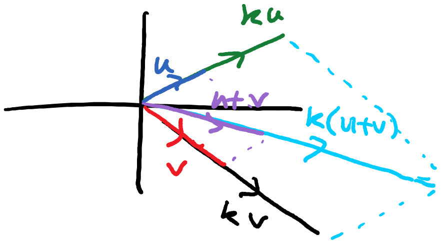
5. If $u$ is in $V$, then there is some object in $V$, called $-u$, with the property that $u+(-u) = 0$ ← special object from 4.



6. If $k$ is a scalar and $u$ is in $V$ then $ku$ is in $V$.

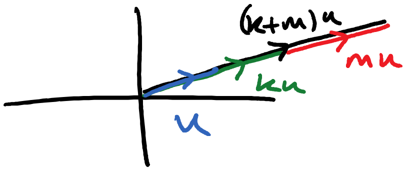


7. If $k$ is a scalar and $u, v$ in $V$ then $k(u+v) = ku + kv$



8. If $k, m$ are scalars and $u$ in $V$, then

$$\underset{\substack{\uparrow\\ \text{addition}\\ \text{of scalars}}}{(k+m)}u = ku \underset{\substack{\uparrow\\ \text{addition in}\\ \text{vector space } V}}{+} mu \; 1$$



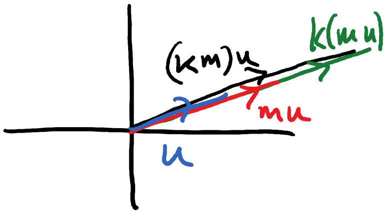
9. If $k, m$ are scalars and $u$ in $V$

then $(km)u = k(mu)$

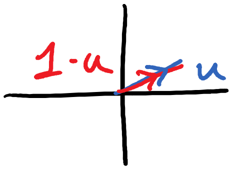
scalar mult.

(and notice this is $m(ku)$

$$= \underbrace{(mk)u = (km)u}_{\text{scalar multiplication}}$$



10. If $u$ is in $V$, then $\underset{\substack{\nearrow\\ \text{scalar}}}{1} \cdot u = u$



We said this in class: the scalars can be reals or can be complex numbers. We fix a choice of scalars: $\mathbb{R}$ or $\mathbb{C}$. Then our vector space is respectively a <u>real v.s.</u> or a <u>complex v.s.</u>

A note about the pictures: special case where $V = \mathbb{R}^2$ and scalars are in $\mathbb{R}$

## Examples

1. $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces.

→ What about the scalars?

$(V=)\mathbb{R}^n$ is a real vector space but NOT a complex vector space

(scalars above "real", scalars below "complex")

e.g. $i(5,6,1) = (5i,6i,i)$

Axiom 6 fails for $\mathbb{R}^3$.

~~For~~ $\mathbb{C}^n$ is a complex vector space

AND a real vector space

(any complex vector space is also a real vector space)

↳ if the axioms hold for all complex # scalars, they automatically hold for all real # scalars, since reals are also complex numbers! From now only just say "complex v.s." (not both).

2. $M_{m\times n}(\mathbb{R})$ — $m\times n$ matrices with real entries

— real v.s.

$M_{m\times n}(\mathbb{C})$ — " " " complex entries

— complex v.s.

3. Zero v.s. $\{0\}$ ; + defined by $0+0=0$

$k0$ " " $k0=0$

is a complex vector space.

We have to define them both to equal 0 as this is the only choice possible.

4. Set of all polynomials of degree at most d

$a_0 + a_1 x + \dots + a_d x^d$ is a real v.s. if $a_i$ real

complex v.s. if $a_i$ complex

(irrelevant where x lives
– we don't evaluate the polynomials)

5. The set of all $\overset{\text{complex}}{\text{real}}$-valued functions defined on $\underset{\mathbb{C}}{\mathbb{R}}$ is a $\underset{\text{complex}}{\text{real}}$ v.s. where $f+g$ is the function defined by $(f+g)(x) = f(x) + g(x)$ for all x and $kf$ is the function defined by $(kf)(x) = kf(x)$ for all k real, for all x

i.e. this is two definitions in one: "real" everywhere $\longrightarrow$ $\overset{\text{complex}}{\text{real}}$ vector space
or "complex" everywhere $\longrightarrow$ complex vector space.

6. The set of infinite sequences $\{a_n\}_{n\in\mathbb{N}}$ of real (respectively complex) numbers is a real ( " " ) v.s. when we define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} = a_0+b_0, a_1+b_1, \dots$$

and

$$k\{a_n\} = \{ka_n\} = ka_0, ka_1, \dots$$

Check out Example 8, p. 188 in textbook.

## Non-examples

highest exponent of the terms with non-zero coefficients.

1. Set of polynomials of degree 3

$$2 - 5x^2 + 6x^3$$
$$1 + 3x + 2x^2 - 6x^3$$

Sum : $3 + 3x - 3x^{\textcircled{2}}$ degree 2 NOT of degree 3 so Axiom 1 fails

Not a real vector space.

2. The set $\mathbb{R}^2$ with + as usual but "scalar multiplication" $k(u_1, u_2) = (ku_1, 0)$

(Why is this NOT a v.s. ? Axiom 10 ...)

**Subspaces** W, a subset of vector space V, is a subspace of V if $W \subseteq V$ and W is a v.S. with the same + and scalar multiplication as in V.

**Theorem** $W \subseteq V$, V a v.S., is a subspace if

① W is not empty.

② If u, v are in W, then u+v is also in W.
( W is "closed under addition")

③ If k scalar, u in W, then ku is in W.
( W is "closed under scalar multiplication")

Examples 1. Any plane through the origin in $\mathbb{R}^3$ (real subspace)

2. $\{0\}$ is a subspace of any vector space

$V$ is a subspace of itself for any vector space $V$

3. If $W_1, \dots, W_n$ are subspaces of $V$, then so is $W_1 \cap \dots \cap W_n$, their intersection.

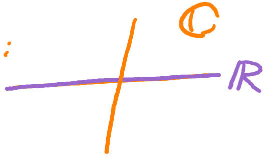
4. If $A$ is a $m \times n$ matrix, then $\{x \in \mathbb{C}^n \mid Ax = 0\}$ is a subspace of $\mathbb{C}^n$

the kernel of A

5. The set of all polynomials of degree at most $d$ with complex coefficients is a subspace of the v.s. of complex valued functions.

6. (Example & Not-an-Example) I was asked during class: Is $\mathbb{R}^n$ a subspace of $\mathbb{C}^n$? Two answers:

(a) As a complex vector space: YES e.g. we think of $\mathbb{R}$ as a line through the origin sitting inside $\mathbb{C}$:



(b) As a real vector space: NO; ③ from the Theorem fails for the same reason that $\mathbb{R}^n$ is not a complex vector space (see #1 Examples of Vector Spaces above: $i\underbrace{(a,b,c)}_{\in \mathbb{R}^3} = \underbrace{(ai,bi,ci)}_{\text{NOT in } \mathbb{R}^3}$).

## Linear independence & spanning

When we have a set of vectors $S = \{v_1, \ldots, v_r\}$ we can make new vectors: linear combinations

$$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r \qquad k_i \text{ all scalars}$$

Collection of all linear combinations like this from $S$ is the span of $S$ $\quad$ span$(S)$

If the only way to get $0$ is by taking all $k_i$ to be $0$, we say $S$ is linearly independent

If there is some solution to the equation

$$k_1 v_1 + \ldots + k_r v_r = 0 \quad \text{other than } k_i = 0 \text{ for all } i = 1, \ldots, r$$

then we say $S$ is linearly dependent.

## Bases & Dimension

$V$ a vector space

$S$ is a set of vectors

$S$ is a basis for $V$ if ① $S$ is linearly independent

② span$(S) = V$ ("$S$ spans $V$")

If $V$ has a finite basis $S$, then it is called finite dimensional & all bases of $V$ have the same size as $S$, called the dimension of $V$.

If $V$ does not have a finite basis, then it is called infinite dimensional.

All definitions (span, lin. ind., basis, dim.) do not depend on whether scalars are real or complex.

1. The set $\{(\overset{\in\mathbb{C}}{1},\overset{\in\mathbb{C}}{0}),\ (\overset{\in\mathbb{C}}{0},\overset{\in\mathbb{C}}{1})\}$ is a basis for $\mathbb{C}^2$ as a complex vector space.

i.e. it is lin. independent and we can obtain every $(z_1, z_2) \in \mathbb{C}^2$ as a linear combination of $(1,0)$ and $(0,1)$ because $(z_1,z_2) = z_1 \cdot (1,0) + z_2 \cdot (0,1)$.

So $\mathbb{C}^2$ has dimension 2 as a complex vector space.

But $\{(1,0),(0,1)\}$ is NOT a basis for $\mathbb{C}^2$ as a real vector space. If we can only use real #s as scalars we cannot e.g. reach $(i,0)$ as a linear combination of $(1,0),(0,1)$.

We need e.g. $\{(1,0),(i,0),(0,1),(0,i)\}$.
(as a basis for $\mathbb{C}^2$ as a real v.s. – where only real scalars are allowed)

So as a real vector space, $\mathbb{C}^2$ is 4-dimensional.
(Just like complex plane $\mathbb{C}$ is 2-dim. as a real vector space.)

We write complex #s as $a+bi$, a linear combination of 1 and $i$ with real scalars!

2. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is the standard basis for $M_2(\mathbb{R})$ as a real vector space and $M_2(\mathbb{C})$ as a complex v.s.

Remember: Theorem (4.5.3 in textbook) (+/−)

$S$ non-empty set of vectors in v.s. $V$

+ 1. If $S$ is lin. independent and $v$ in $V$ lies outside span$(S)$, then span$(S) \cup \{v\}$ is lin. independent.

− 2. If $v$ in $S$, $v = k_1 v_1 + \ldots + k_r v_r$ with $k_i$ scalar and $v_i$ in $S$ then $\text{Span}(S) = \text{span}(\underset{\substack{\uparrow \\ \text{without } v.}}{S \setminus \{v\}})$

Finally coordinates: if $S = \{v_1, \ldots, v_r\}$ is a basis for a v.s. $V$,

then every $v$ in $V$ can be written as $k_1 v_1 + \ldots + k_r v_r$ for some unique choice of scalars $k_1, \ldots, k_r$ ← call these the coordinates of $v$ relative to the basis $S$.

# Dot product

This is all supposed to be very hesitant–
you really shouldn't be thinking of
multiplying vectors! ↓

In (only) one sense we do have a way (sort of) of multiplying vectors: dot product in $\mathbb{R}^n$

If $u = (u_1, \dots, u_n)$
$v = (v_1, \dots, v_n)$ are vectors in $\mathbb{R}^n$, their dot product is

$$u \cdot v = \cancel{(} u_1 v_1 \cancel{,} + \dots \cancel{,} + u_n v_n \cancel{)}$$

We also have a definition of a dot product in $\mathbb{C}^n$. If $u = (u_1, \dots, u_n)$
$v = (v_1, \dots, v_n)$ are vectors in $\mathbb{C}^n$,

the dot product is

$$u \cdot v = \cancel{(} u_1 \overline{v_1} \cancel{,} + \dots \cancel{,} + u_n \overline{v_n} \cancel{)}.$$

T.B.C.!