Last time We covered vector spaces, subspaces, linear (in)dependence, spanning, bases, and ...

Dot / Scalar Product $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)$

In $\mathbb{R}^n$: $u \bullet v = u_1 v_1 + \dots + u_n v_n$

In $\mathbb{C}^n$: $u \bullet v = u_1 \overline{v_1} + \dots + u_n \overline{v_n}$

(These are actually the same, as $z = \overline{z}$ if z is real!)

Properties of the dot/scalar product in $\mathbb{R}^n/\mathbb{C}^n$:

In $\mathbb{R}^n$ $\leftarrow \left\{ \begin{array}{l} u=(u_1,\dots,u_n) \\ v=(v_1,\dots,v_n) \\ w=(w_1,\dots,w_n) \end{array} \right\} \rightarrow$ In $\mathbb{C}^n$

(a) $u \cdot v = v \cdot u$

(a) $u \bullet v = u_1\overline{v_1} + \dots + u_n \overline{v_n}$

$= \overline{\overline{u_1} v_1} + \dots + \overline{\overline{u_n} v_n}$ ← $\overline{u_n}\overline{\overline{v_n}}$

$= \overline{\overline{u_1}v_1 + \dots + \overline{u_n} v_n}$ ← $\overline{z}+\overline{w} = \overline{z+w}$

$= \overline{v \cdot u}$.

(b) $(u+v)\cdot w = u \cdot w + v \cdot w$

(b) $(u+v)\cdot w$

$= (u_1+v_1)\overline{w_1} + \dots + (u_n+v_n)\overline{w_n}$

$= u_1\overline{w_1} + \dots + u_n \overline{w_n}$

$+ v_1\overline{w_1} + \dots + v_n\overline{w_n}$

$= u\cdot w + v \cdot w$

(c) $(ku)\cdot v = k(u\cdot v)$

(Note $u\cdot(kv) \overset{\text{by}}{\underset{(a)}{=}} (kv)\cdot u$

$= k(v\cdot u) = k(u\cdot v)$)

(c) $(ku)\cdot v = (ku_1)\overline{v_1} + \dots + (ku_n)\overline{v_n}$

$= k(u\cdot v)$

(Note $u\cdot(kv) \underset{(a)}{=} \overline{(kv)\cdot u}$

$\underset{(c)}{=} \overline{k(v\cdot u)} = \overline{k}\,\overline{(v\cdot u)}$

$= \overline{k}(u\cdot v)$)

(d) $u \cdot u = u_1^2 + \ldots + u_n^2 \geq 0$

and $u \cdot u = 0$ exactly when $u = 0$

The <u>norm</u> of u in $\mathbb{R}^n$ is

$$\|u\| = \sqrt{u_1^2 + \ldots + u_n^2} = \sqrt{u \cdot u}$$

Once we have a norm, we can define the distance between two vectors:

$$d(u,v) = \|u - v\|$$

Finally: θ angle between u and v is given by $\cos\theta = \dfrac{u \cdot v}{\|u\|\|v\|}$.

(d) $u \cdot u = u_1\bar{u}_1 + \ldots + u_n\bar{u}_n$
$= |u_1|^2 + \ldots + |u_n|^2$
≥ 0

and $u \cdot u = 0$ exactly when $u = 0$.

The <u>norm</u> of u in $\mathbb{C}^n$ is

$$\|u\| = \sqrt{u \cdot u} = \sqrt{|u_1|^2 + \ldots + |u_n|^2}$$

(Note: for $n = 1$ we get $\|u_1\| = |u_1|$.)

Here we define

$$d(u,v) = \|u - v\|.$$

distance between vectors u and v.

Same formula to define angles θ between complex vectors $\dfrac{u \cdot v}{\|u\|\|v\|} = \cos\theta$.

If $n = 1$, u, v in $\mathbb{C}$ we ~~get~~

$$\cancel{\frac{u\bar{v}}{\sqrt{u\bar{u}}\sqrt{v\bar{v}}} = \cos\theta}$$

~~---> to be continued~~

~~(look in textbook for this example)~~

Sorry, I realised later that this is <u>way</u> more complicated, even just in $\mathbb{C}$. It's possible to show that if θ is the difference in arguments of u and v, and ϕ is the angle between them given by the complex dot products, then $\mathrm{Re}(\cos\phi) = \cos\theta$ — DO NOT WORRY ABOUT THIS AT ALL!!! You may definitely ignore this geometric interpretation in $\mathbb{C}$ point of view.

Inner Product Spaces

An inner product is "dot/scalar product" for vector spaces V in general.
We define it by saying that it satisfies all those crucial properties that a dot/scalar product for $\mathbb{R}^n$ or $\mathbb{C}^n$ satisfies:

Def^n

An inner product on a real vector space is a function associating to each pair of vectors u, v a real # $\langle u, v\rangle$ in such a way that the following are all true:

[Axioms for a real inner product:]

1. $\langle u, v\rangle = \langle v, u\rangle$ for all u, v
2. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all u, v, w
3. $\langle ku, w\rangle = k\langle u, w\rangle$ for all u, w vectors and k real #s
4. $\langle u, u\rangle$ is real for all u and $\langle u, u\rangle \geq 0$ for all u and $\langle u, u\rangle = 0$ exactly when $u=0$

A real vector space V with a real inner product is called a real inner product space.

(It's a souped-up real vector space.)

A inner product on a complex vector space is a function associating to each pair of vectors u, v a complex # $\langle u, v\rangle$ in such a way that the following are all true:

[Axioms for a complex inner product:]

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all u, v
2. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all u, v, w
3. $\langle ku, w\rangle = k\langle u, w\rangle$ for all vectors u, w and all complex #s k

   (and notice, for the same reasons as before, $\langle u, kw\rangle = \bar{k}\langle u, w\rangle$)
4. $\langle u, u\rangle \geq 0$ for all u, and $\langle u, u\rangle = 0$ exactly when $u=0$.

A complex vector space with a complex inner product is called a complex inner product space.

Examples

1. The usual dot product on $\mathbb{R}^n$ is a real inner product space (and the complex version is a complex " " ").

2. We can "weight" the usual dot product to preference certain vectors.

e.g. over $\mathbb{C}^2$: $\langle u, v \rangle = (5i)u_1\bar{v}_1 + (6-i)u_2\bar{v}_2$ where $u = (u_1,u_2)$, $v = (v_1,v_2)$

Then $\langle (3+2i, 4+i), (i, i-1) \rangle = (5i)(3+2i)(-i) + (6-i)(4+i)(-i-1) = \ldots$ some complex #

3. We can define a real inner product on the real vector space $M_{m\times n}(\mathbb{R})$ by

$$\langle A, B \rangle = tr(B^TA) \quad \left(= tr(A^TB) = tr(AB^T) \right)$$

e.g. $m=2, n=3$

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 & 0 \\ 0 & -5 & -2 \end{pmatrix}$$

$$B^TA = \begin{pmatrix} 3 & 0 \\ -1 & -5 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 5 \\ 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 15 \\ -11 & -20 & -10 \\ -4 & -8 & -2 \end{pmatrix}$$

$$tr(B^TA) = 3 - 20 - 2 = \underline{\underline{-19}}$$

Let's check ② $\langle A+B, C\rangle = tr(C^T(A+B)) = tr(C^TA + C^TB)$
$= tr(C^TA) + tr(C^TB)$
$= \langle A, C\rangle + \langle B, C\rangle$. ✓

3.′ We can define a complex inner product on $M_{m\times n}(\mathbb{C})$ by $\langle A, B\rangle = tr(\overline{B}^T A)$ where for a matrix $C = [c_{ij}]$, $\overline{C} = [\overline{c_{ij}}]$.

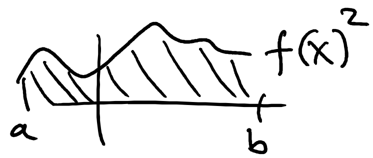
4. There is a way to define a real inner product on the vector space of all real-valued continuous functions on some interval $[a,b]$ using integration:

Take continuous functions f, g and define:

$$\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx$$

Then ① $\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx = \frac{1}{b-a}\int_a^b g(x)f(x)\,dx$
$= \langle g, f\rangle$ ✓

④ $\langle f, f\rangle = \frac{1}{b-a}\int_a^b f(x)^2\,dx \geq 0$



area ≥ 0

$\langle f, f\rangle = 0$ exactly when $f(x)^2 = 0$ i.e. $f(x) = 0$ ✓ as $f(x)^2 \geq 0$ everywhere.

This is a real inner product space.

(There's a complex version $\langle f,g\rangle = \frac{1}{b-a}\int_a^b f(x)\overline{g(x)}\,dx$.)
for $\mathbb{C}$-valued continuous functions on $[a,b]$

5. The vector space of all complex polynomials of degree at most d (called P_d) has an inner product for each choice of $(d+1)$-many distinct complex #s $x_0, \dots, x_d$: define $\langle p,q\rangle = p(x_0)\overline{q(x_0)} + \dots + p(x_d)\overline{q(x_d)}$

Check (the only tricky axiom):

4. For $\langle p,p\rangle = |p(x_0)|^2 + \dots + |p(x_d)|^2 \geq 0$

But when does $\langle p,p\rangle = 0$. For sure when $p=0$.

Anywhere else? When $|p(x_0)|^2 + \dots + |p(x_d)|^2 = 0$

for sure $|p(x_0)| = |p(x_1)| = \dots = |p(x_d)| = 0$

so $p(x_0) = \dots = p(x_d) = 0$.

The only polynomial of degree $\leq d$ with $(d+1)$-many roots is $p=0$.

Magnitude / Modulus / Norm and Distance

Just as we used scalar/dot products to define a norm of a vector and distance/angle between vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ we can use an inner product (if we have one) to define these concepts for a vector space V.

i.e. if V is an inner product space (real or complex) then the <u>norm</u> of a vector u in V is

$$\|u\| = \sqrt{\langle u,u\rangle}$$

← We're OK since $\langle u,u\rangle$ is real and positive, for any u in V.

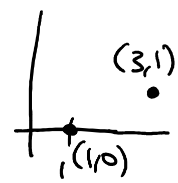
and the <u>distance</u> between two vectors u and v is

$$d(u,v) = \|u - v\| \quad \left(= \sqrt{\langle u-v, u-v\rangle}\right)$$

<u>Example</u>

1. Let's define a weighted inner product on $\mathbb{R}^2$ by $\langle u,v\rangle = 5u_1v_1 + 3u_2v_2$.

   What is $\|(1,0)\|$ and $d((1,0),(3,1))$?



<u>Solution</u> $\|(1,0)\|$

$$= \sqrt{\langle(1,0),(1,0)\rangle} = \sqrt{5.1.1 + \cancel{3.0.0}} = \sqrt{5}$$

$$d((1,0),(3,1)) = \|(-2,-1)\| = \sqrt{\langle(-2,-1),(-2,-1)\rangle}$$

$$= \sqrt{5.(-2)(-2) + 3(-1)(-1)}$$

$$= \sqrt{23}.$$