MATH 2R03 (Linear Algebra II) -- 28 June 2018 -- Lecture 4

Last time: Complex Dot Product &

Inner Product Spaces

a vector space with a function $\langle \cdot , \cdot \rangle$ on pairs of vectors that behaves like the dot product. It gives us

$$\begin{cases} \text{Norm}: & \|u\| = \sqrt{\langle u, u \rangle} \\ \text{Distance}: & d(u,v) = \|u - v\| \end{cases}$$

← notice $\langle u,u \rangle \geq 0$ always real (even in a complex inner product space) — but $\langle u,v \rangle = 0$ not necessarily true if $u \neq v$.

Unit ball: the vectors u with $\|u\| \leq 1$

closed

open unit ball if this is $<$

Additional properties of inner products, norms, distances

Theorem (6.1.2 in real case)

For all u, v, w in innerproduct space V:

(a) $\langle 0, v \rangle = \langle v, 0 \rangle = 0$ ✓

(b) $\langle u, v+w \rangle = \langle u,v \rangle + \langle u,w \rangle$ ✓

(c) $\langle u, v-w \rangle = \langle u,v \rangle - \langle u,w \rangle$ ✓

(d) $\langle u-v, w \rangle = \langle u,w \rangle - \langle v,w \rangle$ ✓

true for complex and real inner product spaces

~~$u(v+w) = uv+uw$~~

(e) $\langle u, kw \rangle = k\langle u, w \rangle$

true for real innerproducts. For complex: $\langle u,kw \rangle = \bar{k}\langle u,w \rangle$ (see L3)

Proof of (a). Let v be in V.

$0 = u - u$ for any vector u.

Pick some u in V. Then $\langle 0, v\rangle = \langle u-u, v\rangle$

$= \langle u+(-u), v\rangle$

$= \langle u, v\rangle + \langle -u, v\rangle$ by Axiom 2

$= \langle u, v\rangle - 1.\langle u, v\rangle$ by Axiom 3

$= 0.$

We know $\langle 0, v\rangle = \langle v, 0\rangle$ by Axiom 1 so $\langle v, 0\rangle = 0$.
So (a) is true. $\square$

Have a go at (b), (c), (d), ...

Theorem 6.1.1 (real case) + 6.2.2

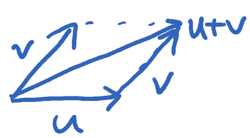
(a) $\|u\| \geq 0$ for all u

$\|u\| = 0$ exactly when $u=0$

(b) $\|ku\| = |k|\|u\|$

(c) $\|u+v\| \leq \|u\| + \|v\|$

(Triangle inequality for norms)

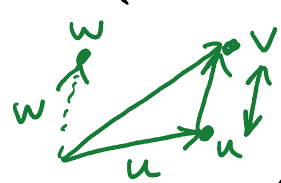


(d) $d(u,v) \geq 0$ for all u, v

$d(u,v) = 0$ exactly when $u = v$

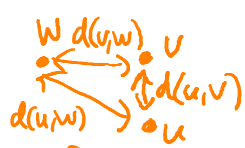
(e) $d(u,v) = d(v,u)$

(f) $d(u,w) \leq d(u,v) + d(v,w)$



(triangle inequality for distances)

Better picture:



(• are endpoints of vectors)

Cauchy - Schwarz Inequality

If u, v are vectors in an inner product space V then

$$|\langle u, v\rangle| \leq \|u\|.\|v\|.$$

(You need this to prove the triangle inequalities.) ← above

Angle & Orthogonality

The dot product in $\mathbb{R}^2$ or $\mathbb{R}^3$ is used to give/ can be defined by:

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

θ is angle between u and v.

So we define the angle between 2 vectors in an inner product space by the formula

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|} \quad (*)$$

In the real case, we need to be between -1 and 1 so this formula makes sense.

Fortunately Cauchy-Schwarz says

$$|\langle u,v\rangle| \leq \|u\|\|v\|$$

So $-\|u\|\|v\| \leq \langle u,v\rangle \leq \|u\|\|v\|$

$$\rightarrow \quad -1 \leq \frac{\langle u,v\rangle}{\|u\|\|v\|} \leq 1 \quad .$$

Phew.

More importantly in $\mathbb{R}^2$ or $\mathbb{R}^3$ we say that 2 vectors are orthogonal (perpendicular) when they are at right angles to each other.
i.e. the angle between them is $\frac{\pi}{2}$.

So here in a general inner product space we say that two vectors u, v are orthogonal if the angle given by the formula (*) is $\frac{\pi}{2}$ i.e. $\cos\theta = 0$

$$\text{So} \quad \frac{\langle u, v\rangle}{\|u\|\|v\|} = 0 \quad \text{i.e.} \quad \boxed{\langle u, v\rangle = 0}$$

(u, v orthogonal/perpendicular means u and v are in some sense as different as could be).

bit vague, not formal

<u>Example</u> Let $C([0,1])$ be the space of all real-valued, continuous functions on $[0,1]$ with its usual inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

Are the functions $x - \frac{1}{2}$, $(x-\frac{1}{2})^2$ orthogonal?

Solution i.e. Is $\langle x-\tfrac{1}{2}, (x-\tfrac{1}{2})^2 \rangle = 0$?

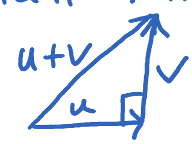
$$= \int_0^1 (x-\tfrac{1}{2})(x-\tfrac{1}{2})^2\,dx = \int_0^1 (x-\tfrac{1}{2})^3\,dx$$

$$= \left[\frac{(x-\tfrac{1}{2})^4}{4}\right]_0^1 = \frac{(\tfrac{1}{2})^4}{4} - \frac{(-\tfrac{1}{2})^4}{4}$$

$$= 0$$ so yes.

Pythagorean Theorem

In $\mathbb{R}^2$, and $\mathbb{R}^3$, we know that if u and v are orthogonal vectors, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.



The same is true in <u>any</u> inner product space (real or complex):

Why? (for real inner product spaces)

$$\|u+v\|^2 = \langle u+v, u+v\rangle \quad (\text{Def}^n \text{ of norm})$$

$$= \langle u,u\rangle + \underbrace{\langle v,u\rangle + \langle u,v\rangle} + \langle v,v\rangle$$

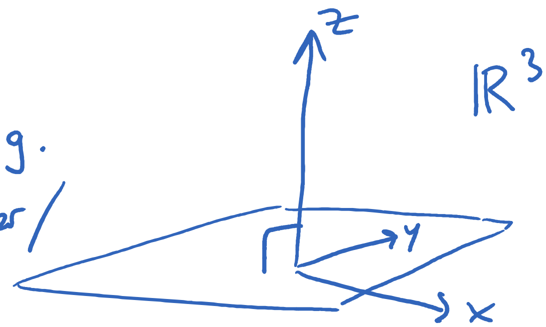
$$= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + 2\cancel{\langle u,v\rangle} + \|v\|^2$$

$\langle u,v\rangle = 0$

$$= \|u\|^2 + \|v\|^2.$$

In $\mathbb{R}^3$ we can think e.g. of z-axis being perpendicular/orthogonal/normal to the subspace of the xy-plane.



We can extend this idea to general inner product spaces i.e. a vector v being orthogonal to a subspace W.

<u>Definition</u> V inner product space, W subspace

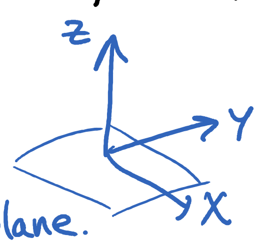
v in V is <u>orthogonal to W</u> if v is orthogonal to w for <u>every</u> w in W i.e. $\langle v, w \rangle = 0$ for every w in W.

Turning this around, we call the set of all vectors v in V which are orthogonal to W the <u>orthogonal complement</u> of W, written $W^\perp$

said "perp."

e.g. z-axis now is W.

Then $W^\perp$ is the xy-plane.



So if you take any vector u in W and any vector v in $W^\perp$ you have $\langle u, v \rangle = 0$.

Exercise 1 Take $M_{3\times 2}(\mathbb{C})$ with the complex inner product we already saw: $\langle A, B\rangle = tr(\overline{B}^T A)$.

Find $d(C,D)$ where $C = \begin{pmatrix} 5 & 0 \\ i & 3 \\ 1+i & 0 \end{pmatrix}$, $D = \begin{pmatrix} 2i & 3-i \\ 0 & 4 \\ 0 & 1-3i \end{pmatrix}$.

Exercise 2 Take $C([0,1])$ with inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$

Find $\|\sin(\pi x)\|$.

Solution 1

$$d(C,D) = \|C-D\|$$
$$= \sqrt{\langle C-D, C-D\rangle}$$
$$= \sqrt{tr(\overline{(C-D)}^T (C-D))}$$

$$C-D = \begin{pmatrix} 5-2i & -3+i \\ i & -1 \\ 1+i & -1+3i \end{pmatrix}$$

$$\overline{(C-D)}^T = \begin{pmatrix} \overline{5-2i} & \overline{i} & \overline{1+i} \\ \overline{-3+i} & \overline{-1} & \overline{-1+3i} \end{pmatrix}$$

$$= \sqrt{tr\begin{pmatrix} |5-2i|^2 + |i|^2 + |1+i|^2 & ? \\ ? & |-3+i|^2 + |-1|^2 + |-1+3i|^2 \end{pmatrix}}$$

? not relevant

$$= \sqrt{|5-2i|^2 + |i|^2 + |1+i|^2 + |-3+i|^2 + |-1|^2 + |-1+3i|^2}$$

$$= \sqrt{\text{sum of } |\text{entries of } C-D|^2}$$

In general if we have matrices $A = [a_{ij}]$ in $M_{m\times n}(\mathbb{C})$, then $B = [b_{ij}]$

$$\langle A, B\rangle = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}\overline{b_{ij}}$$

and
$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m} |a_{ij}|^2}$$

Useful facts!!

Solution 2 Find $\| \sin(\pi x) \|$.

$$\sqrt{\langle \sin(\pi x), \sin(\pi x) \rangle} = \sqrt{\int_0^1 (\sin(\pi x))^2 \, dx}$$

$$= \sqrt{\frac{1}{2}\int_0^1 1 - \cos 2\pi x \, dx}$$

$\sin^2(\pi x) = \frac{1}{2}(1 - \cos 2\pi x)$

$$= \frac{1}{\sqrt{2}} \sqrt{\left[x - \frac{1}{2\pi} \sin 2\pi x\right]_0^1} = \frac{1}{\sqrt{2}}\sqrt{(1 - 0 - 0 + 0)}$$

$$= 1/\sqrt{2}.$$

Notice: not about the values that $\sin(\pi x)$ takes (& how big they are)

Example With the inner product on $M_{22}(\mathbb{R})$ given by $\langle A, B \rangle = tr(B^T A)$, find the angle between $C = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$

Solution Use the useful fact above:

$$\langle C, D \rangle = \sum_{j=1}^{2} \sum_{i=1}^{2} c_{ij} d_{ij} = 0.0 + \sqrt{2}\sqrt{3} + \sqrt{2}.0 + 0.0$$
$$= \sqrt{2}\sqrt{3}.$$

$$\| C \| = \sqrt{\langle C, C \rangle} = \sqrt{0.0 + \sqrt{2}.\sqrt{2} + \sqrt{2}.\sqrt{2} + 0.0}$$
$$= \sqrt{4} = 2$$

$$\| D \| = \sqrt{\langle D, D \rangle} = \sqrt{0.0 + \sqrt{3}.\sqrt{3} + 0.0 + 0.0} = \sqrt{3}$$

Thus the angle θ between C and D is given by

$$\cos\theta = \langle C, D \rangle / \|C\| \, \|D\|$$

$$= \sqrt{2}\sqrt{3} \Big/ 2.\sqrt{3} = 1/\sqrt{2}$$

So $\theta = \cos^{-1}(1/\sqrt{2}) = \pi/4$.

Back to orthogonal complements!

<u>Example</u>

(1) The row space of an $m \times n$ real matrix A and its null space are orthogonal complements in $\mathbb{R}^n$.

(2) The column space of A in (1) is orthogonal to the null space of A^T in $\mathbb{R}^m$.

<u>Proof</u> $A = \begin{pmatrix} \text{---} & r_1 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{pmatrix}$ each $r_i = (r_{i1}, \dots, r_{in})$

Vectors in row space look like

$v = \alpha_1 r_1 + \dots + \alpha_m r_m$, linear comb. ($\alpha_i$ in $\mathbb{R}$)

For a vector w in null space of A, we know $Aw = 0$ i.e. $r_1 \cdot w = r_2 \cdot w = \dots = r_m \cdot w = 0$.

So look at $w \cdot v = \dots = \alpha_1 w \cdot r_1 + \dots + \alpha_m \overset{w \cdot r_m}{\cancel{r_m}}$
$= 0$.

(2) Column space of A is the row space of the transpose A^T (So apply (1) to A^T.)