MATH 2R03 (Linear Algebra II) -- 3 July 2018 -- Lecture 5

Last time

ORTHOGONALITY IN INNER PRODUCT SPACES

Vectors $u, v$ are orthogonal if $\langle u, v \rangle = 0$.

A vector $v$ is orthogonal to a subspace $W$ if $\langle v, u \rangle = 0$ for every vector $u$ in $W$.

$W^{\perp} = \{ v$ in $V$ with $v$ orthogonal to $W \}$ "perp"

The orthogonal complement of $W$.

If $u$ is in $W$, $v$ is in $W^{\perp}$, then $\langle u, v \rangle = 0$.

Some facts about $W^{\perp}$, orthogonal complement.

Theorem (6.2.4 + 6.2.5)

If $W$ is a subspace of an inner product space $V$, then (1) $W^{\perp}$ is itself a subspace of $V$;

(2) the only vector in both $W$ and $W^{\perp}$ is $0$ i.e. $W \cap W^{\perp} = \{0\}$;

(3) If $W$ is finite-dimensional, then $(W^{\perp})^{\perp} = W$.

"Proof" (1) See textbook.

What do we need to show?

(i) $W^{\perp}$ is not empty (try $0$ as $0$ should be in there)

(ii) $W^{\perp}$ is closed under addition

(iii) $W^\perp$ is closed under scalar multiplication by the Theorem earlier.

(2) If $v$ is in $W$ and $W^\perp$ then $\langle \overset{\text{"in } W\text{"}}{v}, \overset{\text{"in } W^\perp\text{"}}{v} \rangle = 0$

so $v = 0$.

(3) Beyond the scope of this crash course. $\square$

## Orthogonal sets & orthogonal (or orthonormal) bases

In contrast to spaces being orthogonal (to each other) we say that a set of vectors <u>$S$ is orthogonal</u> if $\langle u, v \rangle = 0$ for any pair of vectors $u, v$ in $S$.

If $S$ is a basis and is orthogonal (as a set) then $S$ is an <u>orthogonal basis</u>.

If $\|v\| = 1$ for every $v$ in $S$ (and $S$ is orthogonal) then $S$ is <u>orthonormal</u>.

Suppose $\{v_1, \dots, v_n\}$ is an orthogonal basis for an inner product space $V$.

We know that every vector $w$ in $V$ can be written as $w = c_1 v_1 + \dots + c_n v_n$, for scalars $c_i$.

How to find $c_i$?

Recall in $\mathbb{R}^2$: If $w = c_1 v_1 + c_2 v_2$, then

$$w \cdot v_1 = c_1 v_1 \cdot v_1 + c_2 \cancel{v_2 \cdot v_1}$$

0 $v_1, v_2$ orth.

So $c_1 = \dfrac{w \cdot v_1}{\|v_1\|^2}$. Likewise $c_2 = \dfrac{w \cdot v_2}{\|v_2\|^2}$.

So $$w = \overbrace{\left(\frac{w \cdot v_1}{\|v_1\|^2}\right)}^{c_1} v_1 + \overbrace{\left(\frac{w \cdot v_2}{\|v_2\|^2}\right)}^{c_2} v_2$$

If the basis is orthonormal, then

$$w = \underbrace{(w \cdot v_1)}_{c_1} v_1 + \underbrace{(w \cdot v_2)}_{c_2} v_2 .$$

Exactly the same idea works for inner product spaces.

$\{v_1, \dots, v_n\}$ is our orthogonal basis for $V$.

Write $w$ in $V$ as $w = c_1 v_1 + \dots + c_n v_n$.

For each $i = 1, \dots, n$, we can do the following:

$$\langle w, v_i \rangle = c_1 \langle v_1, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle$$

$$= c_i \langle v_i, v_i \rangle$$ (as $\langle v_j, v_i \rangle = 0$ when $i \neq j$ as the basis is orthogonal)

So $c_i = \dfrac{\langle w, v_i \rangle}{\|v_i\|^2}$.

Then

$$w = \overbrace{\left(\frac{\langle w, v_1 \rangle}{\|v_1\|^2}\right)}^{c_1} v_1 + \dots + \overbrace{\left(\frac{\langle w, v_n \rangle}{\|v_n\|^2}\right)}^{c_n} v_n$$

Representation of a vector $w$ in terms of an orthogonal basis $\{v_1, \dots, v_n\}$.

If $\{v_1, \dots, v_n\}$ is an orthonormal basis ($\|v_i\| = 1$ for all $i$) this formula is just

$$w = \overbrace{\langle w, v_1 \rangle}^{c_1} v_1 + \dots + \overbrace{\langle w, v_n \rangle}^{c_n} v_n$$

Representation of a vector $w$ in terms of an orthonormal basis $\{v_1, \dots, v_n\}$.

Example In $P_2([0,1])$, the inner product space of real polynomials of degree at most 2 on the interval $[0,1]$ with the "usual integral inner product" i.e. $\langle p, q \rangle = \int_0^1 p(x) q(x) \, dx$, write

$u(x) = 3x^2$ in terms of the orthogonal basis $\{1, x - 1/2, x^2 - x + 1/6\}$.

Solution Write $v_1 = 1$, $v_2 = x - \frac{1}{2}$, $v_3 = x^2 - x + \frac{1}{6}$.

Notice : We are told $\{v_1, v_2, v_3\}$ is orthogonal so we do not need to check this!

The formula says:

$$3x^2 = u = \left(\frac{\langle u, v_1 \rangle}{\|v_1\|^2}\right) v_1 + \left(\frac{\langle u, v_2 \rangle}{\|v_2\|^2}\right) v_2 + \left(\frac{\langle u, v_3 \rangle}{\|v_3\|^2}\right) v_3.$$

So we need to find $\langle u, v_1 \rangle$, $\langle u, v_2 \rangle$, $\langle u, v_3 \rangle$,
$\|v_1\|^2$, $\|v_2\|^2$, $\|v_3\|^2$.

$$\langle u, v_1 \rangle = \langle 3x^2, 1 \rangle = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = \underline{\underline{1}}$$

$$\langle u, v_2 \rangle = \langle 3x^2, x - \tfrac{1}{2} \rangle = \int_0^1 3x^2(x - \tfrac{1}{2})\,dx = \int_0^1 3x^3 - \tfrac{3}{2}x^2\,dx$$

$$= \left[\tfrac{3}{4}x^4 - \tfrac{1}{2}x^3\right]_0^1 = \tfrac{3}{4} - \tfrac{1}{2} = \underline{\underline{\tfrac{1}{4}}}$$

$$\|v_1\|^2 = \underline{\underline{1}}$$

$$\|v_2\|^2 = \langle v_2, v_2 \rangle = \langle x - \tfrac{1}{2}, x - \tfrac{1}{2} \rangle = \int_0^1 (x - \tfrac{1}{2})^2\,dx$$

$$= \int_0^1 x^2 - x + \tfrac{1}{4}\,dx$$

$$= \left[x^3/3 - x^2/2 + x/4\right]_0^1 = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4}$$

$$= \underline{\underline{\tfrac{1}{12}}}.$$

$$\langle u, v_3 \rangle = \int_0^1 (3x^2)(x^2 - x + \tfrac{1}{6})\,dx = \int_0^1 3x^4 - 3x^3 + \tfrac{x^2}{2}\,dx$$

$$= \left[\tfrac{3x^5}{5} - \tfrac{3}{4}x^4 + \tfrac{x^3}{6}\right]_0^1 = \tfrac{3}{5} - \tfrac{3}{4} + \tfrac{1}{6}$$

$$= \frac{36 - 45 + 10}{60} = \underline{\tfrac{1}{60}}$$

$$\|v_3\|^2 = \langle v_3, v_3 \rangle = \int_0^1 (x^2 - x + \tfrac{1}{6})^2\,dx$$

$$= \int_0^1 x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{1}{3}x + \tfrac{1}{36}\,dx$$

$$= \left[ \frac{x^5}{5} - \frac{x^4}{2} + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{36-90+80-30+5}{180}$$

$$= \underline{1/180}.$$

So putting it all together:

$$u = \left(\frac{\langle u, v_1\rangle}{\|v_1\|^2}\right)v_1 + \left(\frac{\langle u, v_2\rangle}{\|v_2\|^2}\right)v_2 + \left(\frac{\langle u, v_3\rangle}{\|v_3\|^2}\right)v_3$$

$$= \frac{1}{1}\cdot 1 + \frac{1/4}{1/12}\left(x - \tfrac{1}{2}\right) + \frac{1/60}{1/180}\left(x^2 - x + \tfrac{1}{6}\right)$$

$$= 1 + 3\left(x - \tfrac{1}{2}\right) + 3\left(x^2 - x + \tfrac{1}{6}\right).$$

can check this is $3x^2$ by expanding.

**Theorem** In an inner product space, any orthogonal set is linearly independent.

**Proof** Let $S = \{v_1, \dots, v_k\}$ be an orthogonal set and let $0 = c_1v_1 + \dots + c_kv_k$ for some scalars $c_i$.

Want to show $c_i = 0$ for all $i$.

Well, what are the $c_i$?

$$c_i = \frac{\langle 0, v_i\rangle}{\|v_i\|^2} \quad \text{(from the formula above)}$$

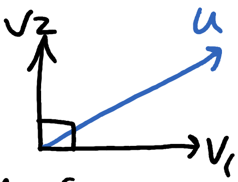
$$= 0.$$

So $S$ is linearly independent. $\square$

## Projection

Back to $\mathbb{R}^2$:

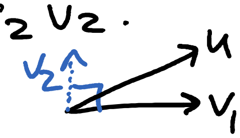
If $v_1$ and $v_2$ are orthogonal, any vector $u$ has a "component in direction $v_1$" and a " " " $v_2$".




i.e. $u = c_1 v_1 + c_2 v_2$ (we found formulae for $c_1, c_2$ above)

Also: <u>Projection Theorem for $\mathbb{R}^2$</u>

If we have any vectors $u$ and $v_1$, there is a vector $v_2$ orthogonal to $v_1$ such that $u$ can be written in a unique way as $u = c_1 v_1 + c_2 v_2$.




<u>Notice</u>: "$v_2$ orthogonal to $v_1$" is the same as saying "$v_2$ is orthogonal to $\text{span}(\{v_1\})$" or "$v_2$ is in $(\text{span}\{v_1\})^\perp$."

<u>Projection Theorem</u> For any vector $u$ in an inner product space $V$ and for any finite-dimensional subspace $W$ (of $V$), $u$ can be written in a unique way as

$$u = \underbrace{w_1}_{w_1 \text{ is in } W} + w_2 \quad \text{where } w_2 \text{ is in } W^\perp.$$

$w_1$ is denoted $\text{proj}_W u$, the projection of $u$ onto $W$

$w_2$ is denoted $\text{proj}_{W^\perp} u$, the projection of $u$ onto $W^\perp$.

How to find $\mathrm{proj}_W u$ and $\mathrm{proj}_{W^\perp} u$?

We can do this if we have an orthogonal basis for $W$. Call it $\{v_1, \dots, v_k\}$.

Since $\mathrm{proj}_W u$ is in $W$ and $\{v_1, \dots, v_k\}$ is an orthogonal basis for $W$ we can use the formula from earlier:

$$\mathrm{proj}_W u = \left(\frac{\langle \mathrm{proj}_W u, v_1 \rangle}{\|v_1\|^2}\right) v_1 + \dots + \left(\frac{\langle \mathrm{proj}_W u, v_k \rangle}{\|v_n\|^2}\right) v_k$$

Notice

$$\begin{aligned} \langle u, v_i \rangle &= \langle \mathrm{proj}_W u + \mathrm{proj}_{W^\perp} u, v_i \rangle \\ &= \langle \mathrm{proj}_W u, v_i \rangle + \underbrace{\langle \underbrace{\mathrm{proj}_{W^\perp} u}_{\text{in } W^\perp}, \underbrace{v_i}_{\text{in } W} \rangle}_{0} \end{aligned}$$

Substituting in we get

$$\boxed{\mathrm{proj}_W u = \left(\frac{\langle u, v_1 \rangle}{\|v_1\|^2}\right) v_1 + \dots + \left(\frac{\langle u, v_k \rangle}{\|v_k\|^2}\right) v_k.}$$

Component of $u$ in the subspace $W$.

And $\boxed{\mathrm{proj}_{W^\perp} u = u - \mathrm{proj}_W u}$ ← Component of $u$ in the subspace $W^\perp$

<u>Example</u> In $M_{22}(\mathbb{R})$ with its usual inner product $\langle A, B \rangle = \mathrm{tr}(B^T A)$, write

$u = \begin{pmatrix} 3 & -5 \\ 2 & 1 \end{pmatrix}$ in terms of its projections onto $W$ and $W^\perp$, where $W = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \right\}$.

Solution First we need an orthogonal basis for $W$. Let's try $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \right\}$

(we already know it spans $W$ ✓ )

Check orthogonality next:

$$\left\langle \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \right\rangle = 1.1 + 0.5 + (-2)(-1) + 3.(-1)$$
$$= 1 + 2 - 3 = 0. ✓$$

Then we get linear independence for free by the earlier observation. ✓

So $\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}}_{v_2} \right\}$ is an orthogonal basis for $W$.

$$\text{proj}_W u = \left( \frac{\langle u, v_1 \rangle}{\|v_1\|^2} \right) v_1 + \left( \frac{\langle u, v_2 \rangle}{\|v_2\|^2} \right) v_2$$

$$= \frac{\left\langle \begin{pmatrix} 3 & -5 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} + \frac{\left\langle \begin{pmatrix} 3 & -5 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}$$

$$= \frac{3 - 4 + 3}{1 + 4 + 9} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} + \frac{3 - 25 - 2 - 1}{1 + 25 + 1 + 1} \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} - \frac{25}{28}\begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} -21 & -125 \\ 17 & 37 \end{pmatrix}.$$

$$\begin{aligned} \mathrm{proj}_{W^\perp} u &= u - \mathrm{proj}_W u \\ &= \begin{pmatrix} 3 & -5 \\ 2 & 1 \end{pmatrix} - \frac{1}{28}\begin{pmatrix} -21 & -125 \\ 17 & 37 \end{pmatrix} \\ &= \text{something. (a single matrix)}. \end{aligned}$$

$$\begin{pmatrix} 15/4 & -15/28 \\ 39/28 & -9/28 \end{pmatrix}$$

The argument above used an orthogonal basis for W. Can we always find such a basis for any finite-dimensional inner product space?

<u>Yes</u>. Every vector space (or subspace) has a basis.

We can transform any basis into an orthogonal basis using the <u>Gram-Schmidt Process</u>.

The GS Process for finite-dimensional inner product spaces goes exactly like the procedure for $\mathbb{R}^2$ / $\mathbb{R}^3$ (with $\langle \cdot , \cdot \rangle$ in place of $\cdot$ ).

Q: Isn't the GS Process about orthonormal bases?

First note you can transform any orthogonal basis into an orthonormal one by <u>normalizing</u> each vector (divide by its own modulus/norm):

$$\{v_1, \ldots, v_k\} \longrightarrow \left\{ \frac{v_1}{\|v_1\|}, \ldots, \frac{v_k}{\|v_k\|} \right\}.$$

(They do the GS Process this way in the textbook.)

But, yes, there are reasons why orthonormality is very useful to have as we go along (e.g. makes formulae easier) so we describe the GS process to go straight to an orthonormal basis (we have to remember to normalize at every step).

# Gram-Schmidt Process

1. Start with any basis for $V$.
   Call it $\{u_1, \dots, u_n\}$.

2. Pick $u_1$ and normalize it; call this $v_1$
   i.e. $v_1 = \dfrac{u_1}{\|u_1\|}$. (Notice span $\{u_1\}$ = span $\{v_1\}$.)

3. Pick $u_2$ and write it as
   $$u_2 = \mathrm{proj}_{W_1} u_2 + \mathrm{proj}_{W_1^\perp} u_2$$ where
   $W_1 = \mathrm{span}\{v_1\}$.

   Really we're interested in this new contribution ($\mathrm{proj}_{W_1} u_2$ lies in ).

   Rearrange: $\mathrm{proj}_{W_1^\perp} u_2 = u_2 - \mathrm{proj}_{W_1} u_2$
   $= u_2 - \langle u_2, v_1 \rangle v_1$

   Take this and normalize; call that $v_2$.
   $$\left( \frac{u_2 - \langle u_2, v_1\rangle v_1}{\|u_2 - \langle u_2, v_1 \rangle v_1\|} = v_2 \right).$$

   Notice that span $\{v_1, v_2\}$ = span $\{u_1, u_2\}$
   and $v_2$ orthogonal to $v_1$.

4. Pick $u_3$ and write it as
   $$u_3 = \mathrm{proj}_{W_2} u_3 + \mathrm{proj}_{W_2^\perp} u_3$$
   where $W_2 = \mathrm{span}\{v_1, v_2\}$.

   We're interested in this:

$$\mathrm{proj}_{W_2^\perp} u_3 = u_3 - \mathrm{proj}_{W_2} u_3$$
$$= u_3 - (\langle u_3, v_1\rangle v_1 + \langle u_3, v_2\rangle v_2)$$

Normalize this; call it $v_3$.

$$v_3 = \frac{u_3 - (\langle u_3, v_1\rangle v_1 + \langle u_3, v_2\rangle v_2)}{\| u_3 - (\langle u_3, v_1\rangle v_1 + \langle u_3, v_2\rangle v_2 \|}$$

Notice that $\mathrm{span}\{u_1, u_2, u_3\} = \mathrm{span}\{v_1, v_2, v_3\}$ and $v_i$ are mutually orthogonal.

(5.) Continue picking $u_i$s and making $v_i$s in this way until you run out of $u_i$s.

This gives an orthonormal basis $\{v_1, \dots, v_n\}$ for $V$.

Example Look at $P_2([0,1])$, space of polynomials on $[0,1]$ of degree $\leq 2$, with the integral inner product. Find an orthonormal basis for $P_2([0,1])$ starting with the basis $\{1, x, x^2\}$.