Exercise from Lecture 6 on QR Decomposition

Exercise Find a QR Decomposition of $A = \begin{pmatrix} 3 & 5 \\ 4 & 15 \\ 0 & 12 \end{pmatrix}$.

Solution First notice that the columns of A form a linearly independent set. So we run the Gram-Schmidt Process to obtain an orthonormal basis for $\operatorname{Span}\left\{ \underbrace{\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 5 \\ 15 \\ 12 \end{pmatrix}}_{u_2} \right\}$.

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{(3 \;\; 4 \;\; 0)^T}{\|(3 \;\; 4 \;\; 0)^T\|} = \left(\tfrac{3}{5} \;\; \tfrac{4}{5} \;\; 0\right)^T.$$

Find $u_2 - \langle u_2, v_1 \rangle v_1 = (5 \;\; 15 \;\; 12)^T - \langle (5 \; 15 \; 12)^T, (\tfrac{3}{5} \; \tfrac{4}{5} \; 0)^T \rangle (\tfrac{3}{5} \; \tfrac{4}{5} \; 0)^T$

$$= (5 \;\; 15 \;\; 12)^T - (3+12)\left(\tfrac{3}{5} \;\; \tfrac{4}{5} \;\; 0\right)^T$$
$$= (5 \;\; 15 \;\; 12)^T - (9 \;\; 12 \;\; 0)^T = (-4 \;\; 3 \;\; 12)^T$$

This has norm $\|(-4 \;\; 3 \;\; 12)\|^T = \sqrt{16+9+144} = \sqrt{169} = 13$.

So $v_2 = \dfrac{(-4 \;\; 3 \;\; 12)^T}{13} = \left(-\tfrac{4}{13} \;\; \tfrac{3}{13} \;\; \tfrac{12}{13}\right)^T$.

We know $A = QR$ where $Q = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 3/5 & -4/13 \\ 4/5 & 3/13 \\ 0 & 12/13 \end{pmatrix}$

and $R = \begin{pmatrix} u_1 \cdot v_1 & u_2 \cdot v_1 \\ 0 & u_2 \cdot v_2 \end{pmatrix}$.

2/ So compute $u_1 \cdot v_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = \frac{9+16}{5} = 5$

$$u_2 \cdot v_1 = \begin{pmatrix} 5 \\ 15 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = 3 + 12 = 15$$

$$u_2 \cdot v_2 = \begin{pmatrix} 5 \\ 15 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} -4/13 \\ 3/13 \\ 12/13 \end{pmatrix} = \frac{-20+45+144}{13} = \frac{169}{13} = 13.$$

So $R = \begin{pmatrix} 5 & 15 \\ 0 & 13 \end{pmatrix}.$

Thus we can write $A = \begin{pmatrix} 3 & 5 \\ 4 & 15 \\ 0 & 12 \end{pmatrix}$ as

$$QR = \begin{pmatrix} 3/5 & -4/13 \\ 4/5 & 3/13 \\ 0 & 12/13 \end{pmatrix} \begin{pmatrix} 5 & 15 \\ 0 & 13 \end{pmatrix}$$

(check that this product really is A !)