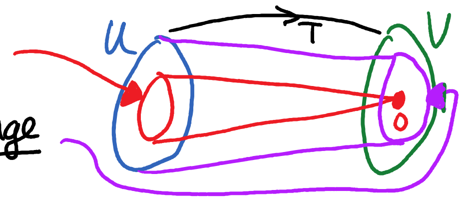
Last time   LINEAR TRANSFORMATIONS

Maps $T: \mathcal{U} \longrightarrow V$ between vector spaces $\mathcal{U}$ and V that respect the additions & scalar multiplications of the vector spaces:

$$\begin{cases} ① \ T(u_1+u_2) = T(u_1)+T(u_2) \text{ for all } u_1, u_2 \text{ in } \mathcal{U} \\ ② \ T((cu)) = (cT(u)) \text{ for all } u \text{ in } \mathcal{U} \text{ \& scalars } c. \end{cases}$$

We have $\ker(T) = \{u \in \mathcal{U} \mid T(u)=0\}$ ← kernel

$R(T) = \{v \in V \mid T(u) = v \text{ for some } u \text{ in } \mathcal{U}\}$ ← range



Theorem If $T: \mathcal{U} \to V$ is a linear transformation, then $\ker(T)$ is a subspace of $\mathcal{U}$ and $R(T)$ is a subspace of V.

Proof $\ker(T) \to$ left as Exercise.

For $R(T)$ we want to show (1) $R(T)$ not empty
(2) $R(T)$ closed under addition (3) $R(T)$ closed under scalar multiplication.

For (1) we pick a vector in V and show it lives in $R(T)$.
We pick the zero vector in V: $T(0)=0$ so 0 is in $R(T)$.

(2) If v, y in $R(T)$ then there are some u and w in $\mathcal{U}$ with
$T(u)=v$ and $T(w)=y$.
Then $v+y = T(u)+T(w) = T(u+w) \in R(T)$.

(3) If v in $R(T)$ and k a scalar, then there is a u in $\mathcal{U}$ with $T(u)=v$. So $kv = kT(u) = T(ku) \in R(T)$. //

If $T: U \to V$ is a linear transformation, and

if $\ker(T)$ is finite dimensional, then $\dim(\ker(T))$ is called the <u>nullity of T</u>,

& if $R(T)$ is finite dimensional, then $\dim(R(T))$ is called the <u>rank of T</u>.

Notice that if $T_A: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix trans.

$T_A(x) = Ax$, then $\text{nullity}(T_A) = \text{nullity}(A)$

[= dim. of nullspace of A]

& $\text{rank}(T_A) = \text{rank}(A)$

[= dim. of column space of A]

<u>Rank-Nullity Theorem</u> If $T: U \to V$ is a linear trans. and U is finite dimensional, then $R(T)$ is finite dim.

& $\boxed{\text{rank}(T) + \text{nullity}(T) = \dim(U).}$

So if $T_A: \mathbb{R}^n \to \mathbb{R}^m$ is given $T_A(x) = Ax$, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

<u>Examples</u> (A)

(B) $I: U \to U$, $I(u) = u$ for all u $\quad \text{nullity}(I) = \dim(\{0\}) = 0$

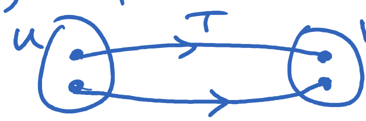
$\text{rank}(I) = \dim(U)$

(c) $T(u) = 0$ for all u: $\text{nullity}(T) = \dim(U)$

$\text{rank}(T) = \dim(\{0_V\}) = 0$

(D) Projection in $\mathbb{R}^3$ onto (x,y)-plane : $T(x,y,z) = (x,y,0)$

$\left.\begin{array}{l} \text{nullity}(T) = \dim(z\text{-axis}) = 1 \\ \text{rank}(T) = \dim((x,y)\text{-plane}) = 2 \end{array}\right\}$ $1+2 = \dim(\mathbb{R}^3)$

Just as with matrix transformations a linear transformation is one-to-one (1-1) if it sends distinct vectors to distinct vectors 

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<u>Theorem</u> If $T: U \to V$ is a linear trans., then the following are equivalent (i) T is one-to-one;

(ii) $\ker(T) = \{0\}$.

<u>Proof</u> <u>(i) $\Rightarrow$ (ii)</u> is clear from $T(0) = 0$.

<u>(ii) $\Rightarrow$ (i)</u> If $\ker(T) = \{0\}$ and $x \neq y$ in U, then suppose (for a contradiction) that $T(x) = T(y)$

Then $T(x-y) = 0$ (by ① & ②) so $x - y \in \ker(T)$.

So $x - y = 0$ i.e. $x = y$ ↯ ※ ✕. ← Contradiction!

So (our supposition was wrong) $T(x) \neq T(y)$ so T 1-1.

A linear transformation $T: U \to V$ is <u>onto</u> (or <u>onto V</u>) if $R(T) = V$ (every vector in V is the image under T of some vector in U).

Theorem If U & V are finite-dimensional & $\dim(U) = \dim(V)$, then also equivalent to (i) & (ii) from Theorem above is

(iii) T is onto.

Proof (ii) $\Rightarrow$ (iii) Suppose $\ker(T) = \{0\}$. By Rank-Nullity Theorem

$$\dim(V) = \dim(U) = \dim(\ker(T)) + \dim(R(T))$$
$$= 0 + \dim(R(T))$$

$R(T)$ subspace of V of the same dim. $\Rightarrow R(T) = V$.

i.e. T is onto.

(That proves (ii) $\Rightarrow$ (iii).)

(iii) $\Rightarrow$ (ii) If $R(T) = V$, then Rank-Nullity Theorem gives

$$\dim(V) = \dim(\ker(T)) + \dim(V) \text{ so}$$
$$\dim(\ker(T)) = 0 \quad \text{so } \ker(T) = \{0\}.$$

Inverses of Linear Transformations

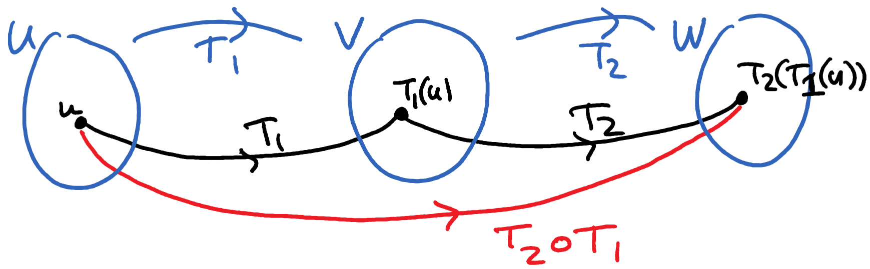
If $T: U \to U$ is 1-1, then it makes sense to define $T^{-1}: R(T) \to U$, "the inverse of T".

If U, V finite dim. & $\dim(U) = \dim(V)$, then T onto and we can write $T^{-1}: V \to U$.

Exercise: T^{-1} is a linear trans. for any linear trans. T.

Compositions of Linear Transformations

If $T_1 : U \longrightarrow V$, $T_2 : V \longrightarrow W$ are linear trans. there is a linear trans. written $T_2 \circ T_1 : U \longrightarrow W$ which satisfies $(T_2 \circ T_1)(u) = T_2(T_1(u))$.



Examples (A) If $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is $T_A(x) = Ax$

$T_B : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ is $T_B(y) = By$

then $T_B \circ T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is given by

$$(T_B \circ T_A)(x) = T_B(T_A(x)) = B(Ax)$$
$$= (BA)x$$

(G) If $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is projection onto (x,y)-plane

$T_1(x,y,z) = (x,y,0)$ and

$T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is projection onto (x,z)-plane

$T_2(x,y,z) = (x,0,z)$,

then $T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is $(T_2 \circ T_1)(x,y,z) = T_2(x,y,0)$
$= (x,0,0)$

Projection onto x-axis (a single operator)

(Exercise: we said that all linear trans. on $\mathbb{R}^n$ can be given as matrix trans. So what are the corresponding matrices for these 3 projections?)

(H) $D(f) = f'$. Then $(D \circ D)(f) = f''$

$\hookrightarrow D(D(f)) = D(f')$.

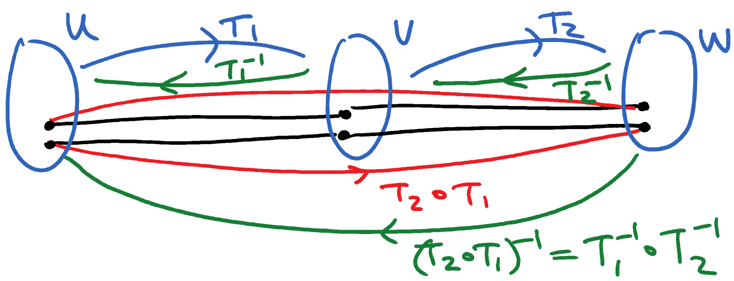
<u>Theorem</u> If $T: U \to V$ is a 1-1 linear trans. then

$I = T \circ T^{-1} : R(T) \longrightarrow R(T)$ and $I = T^{-1} \circ T : U \to U$

(i.e. each is equal to the identity on the appropriate domain)

If $T_1 : U \to V$ and $T_2 : V \to W$ are 1-1 linear trans. then (a) $T_2 \circ T_1$ is 1-1

(b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$



<u>Non-Example</u> If $J(f) = \int_0^x f(t)\,dt$, then J is <u>NOT</u> the inverse of $D : F \longmapsto F'$ as J picks out a <u>particular</u> antiderivative of f : $J(f) = F(x) - F(0)$ for all x

BUT : $D(F(x) + c) = f(x)$ so we do not necessarily have $J(D(g)) = g$ e.g. $J(D(x+1)) = x$ $\neq x + 1$.

A general idea about correspondence between vector spaces : <u>isomorphism</u>.

If $T: U \to V$ is a linear trans. that is both 1-1 and onto we call T a <u>bijection</u> or an <u>isomorphism</u> (between U and V).

Aside on the bijection v. isomorphism distinction:

More generally in mathematics, "bijection" = 1-1 + onto, while "isomorphism" is a special kind of bijection that preserves inherent structural properties of the domain. Since a linear transformation is already (by def^n.) known to preserve addition & scalar multiplication, the inherent structure of a vector space, a linear transformation which is a bijection is also an isomorphism.

So if the is an isomorphism between U and V we say that U and V are <u>isomorphic</u>. This means they look the same (up to relabelling everything) as vector spaces.

For (non-example) if U, V have $\dim(U) \neq \dim(V)$, then U & V are NOT isomorphic.

Because "U & V isomorphic" really means "there is a 1-1 correspondence between basis elements".

In particular, any real n-dim. vector space V is isomorphic to $\mathbb{R}^n$: use coordinates relative to a basis of V:

Fix a basis $S = \{v_1, \dots, v_n\}$ for V.

If v is a vector in V, it can be written uniquely as $v = a_1 v_1 + \dots + a_n v_n$ for <u>coordinates</u> (scalars) $a_1, \dots, a_n$ in $\mathbb{R}$.

So there is a 1-1 correspondence between elements v in V and n-tuples of coordinates $[v]_S = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$.

Claim The map $T: V \longrightarrow \mathbb{R}^n$ given by $T(v) = [v]_S$ is an isomorphism.

Need to check: (1) T is a lin. trans.
(2) T is 1-1
(3) T is onto.

For (1) $T(v+w) = T(\overbrace{a_1v_1 + \ldots + a_nv_n}^{v} + \overbrace{b_1v_1 + \ldots + b_nv_n}^{w})$ where $[w]_S = (b_1, \ldots, b_n)$

$$= T((a_1+b_1)v_1 + \ldots + (a_n+b_n)v_n)$$
$$= (a_1+b_1, \ldots, a_n+b_n)$$
$$= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) = T(v) + T(w).$$

& $T(cv) = T(ca_1v_1 + \ldots + ca_nv_n)$
$$= (ca_1, \ldots, ca_n) = c(a_1, \ldots, a_n) = cT(v).$$

So T is a linear trans.

For (2) if $v \neq w$, then v and w have distinct representations in terms of the basis vectors in S so distinct sets of coordinates.
So T is 1-1.

For (3) for any n-tuple $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ the vector $u = c_1v_1 + \ldots + c_nv_n$ lies in V. (Then $T(u) = (c_1, \ldots, c_n)$.) So T is onto.

So real vector spaces of dimension n "look like" (are isomorphic to) $\mathbb{R}^n$, and complex vector spaces of dimension n "look like" (are isomorphic to $\mathbb{C}^n$.

Exactly as above but coordinate n-tuples $[v]_S$ lie in $\mathbb{C}^n$.

So any 2 vector spaces U and V are isomorphic if $\dim(U) = \dim(V)$.
(Fix bases S_U for U and S_V for V. Send u to the vector v in U with the "same coordinates" i.e. $[v]_{S_V} = [u]_{S_U}$.)

<u>Examples</u> (1) $T : P_3 \longrightarrow \mathbb{R}^4$ given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0, a_1, a_2, a_3)$$

is an isomorphism.

(2) $T: M_{3\times 2}(\mathbb{C}) \longrightarrow \mathbb{C}^6$ given by

$$T\left(\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\right) = (a, b, c, d, e, f)$$ is an isomorphism.

The connection between n-dim. vector spaces and $\mathbb{R}^n$ (or $\mathbb{C}^n$) extends beyond just coordinates:

<u>Example</u> Let $D : P_2 \longrightarrow P_1$ be the differentiation transformation $D(ax^2 + bx + c) = 2ax + b$

We have isomorphisms

$T_2 : P_2 \to \mathbb{R}^3$; $T_2(a_2x^2 + a_1x + a_0) = (a_2, a_1, a_0)$

$T_1 : P_1 \longrightarrow \mathbb{R}^2$; $T_1(b_1x + b_0) = (b_1, b_0)$

(using coordinates relative to the bases $\{x^2, x, 1\}$ for P_2 and $\{x, 1\}$ for P_1).

$$\begin{tikzcd} P_2 \arrow[r, "D"] \arrow[d, "T_2"'] & P_1 \arrow[d, "T_1"] \\ \mathbb{R}^3 \arrow[r, dashed, "?"] & \mathbb{R}^2 \end{tikzcd}$$

There is a (matrix) transformation $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which corresponds to $D: P_2 \longrightarrow P_1$.

T_A should send (a, b, c) to $(2a, b)$.

This is given by $A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix} \longrightarrow A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

__Big Idea__ Using isomorphisms with $\mathbb{R}^n$ (or $\mathbb{C}^n$) we can represent ANY linear transformation $T: U \to V$ between finite dimensional vector spaces U and V as a matrix transformation of coordinates.

$\to$ & thereby translate problems about computing linear transformations into performing matrix computations.

$$\begin{tikzcd}
U \arrow[r, "T"] \arrow[d] & V \arrow[d] \\
\mathbb{R}^n \arrow[r] & \mathbb{R}^m
\end{tikzcd}$$

Let's say $T: U \to V$ is linear
$\dim(U) = n$
$\dim(V) = m$.
Fix bases B_U for U & B_V for V.

Use these to get isomorphisms $U \to \mathbb{R}^n$, $V \to \mathbb{R}^m$ (coordinates relative to B_U and B_V respectively).

We construct an $m \times n$ matrix A ~~so that~~ $(T_A(x) = Ax)$.

It needs to satisfy $A[u]_{B_U} = [T(u)]_{B_V}$

$[u]_{B_U}$: coords of u rel. to B_U

$[T(u)]_{B_V}$: coords of $T(u)$ rel. to B_V.