Last time: ISOMORPHISMS $T: U \to V$ linear, 1-1, onto

(U and V look like relabelled copies of one another.)

U a real (complex) vector space, $\dim(U) = n$:

U is isomorphic to $\mathbb{R}^n$ ($\mathbb{C}^n$).

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[d] & V \arrow[d] \\ \mathbb{R}^n \arrow[r, dashed, "?"'] & \mathbb{R}^m \end{tikzcd}$$

$\cdot = T_A$?

$B_U = \{u_1, \ldots, u_n\}$ — basis for U

$B_V = \{v_1, \ldots, v_m\}$ — basis for V

Wants A to send coordinates (in $\mathbb{R}^n$) with respect to B_U to coordinates w.r.t. B_V. (in $\mathbb{R}^m$)

We construct $m \times n$ matrix A with: $A[u]_{B_U} = [T(u)]_{B_V}$

A is called the matrix for T with respect to the bases B_U and B_V, written $[T]_{B_V, B_U}$ ← watch order

So we (would) have $[T]_{B_V, B_U} [u]_{B_U} = [T(u)]_{B_V}$. for every u in U.

How to find $[T]_{B_V, B_U}$?

We need in particular that

$$[T]_{B_V, B_U} [u_i]_{B_U} = [T(u_i)]_{B_V}$$ for every basis vector u_i in B_U

(where $B_U = \{u_1, \ldots, u_n\}$).

But $[u_i]_{B_u} = e_i = (0 \dots \underset{\uparrow \text{ith place}}{1} \dots 0)$

So we ~~get~~ want $[T]_{B_V, B_u} [e_i] = [T(u_i)]_{B_V}$

We said that for any matrix M, Me_i is the ith column of M, so what we've found is that the ith column of $[T]_{B_V, B_u}$ is $[T(u_i)]_{B_V}$

In other words $\boxed{[T]_{B_V, B_u} = \begin{pmatrix} [T(u_1)]_{B_V} & \cdots & [T(u_n)]_{B_V} \end{pmatrix}}$.

<u>Example</u> above revisited

$D : P_2 \longrightarrow P_1$; $D(ax^2+bx+c) = 2ax+b$

Basis for $P_2 = \{x^2, x, 1\}$, Basis for $P_1 = \{x, 1\}$

$$D(x^2) = 2x = 2 \cdot x + 0\,1$$
$$D(x) = 1 = 0\,x + 1\,1$$
$$D(1) = 0 = 0 \cdot x + 0\,1$$

So $$A = \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ 2x & 1 & 0 \end{matrix} \end{array}$$

Example $I : U \longrightarrow U$ is identity operator & $B_U = \{u_1, \dots, u_n\}$ is a basis for U (on both sides)

Here : $[u_i]_{B_U} = e_i$ for each i so our matrix for I is

$$\begin{pmatrix} [I(u_1)]_{B_U} & \cdots & [I(u_n)]_{B_U} \end{pmatrix} = \begin{pmatrix} [u_1]_{B_U} & \cdots & [u_n]_{B_U} \end{pmatrix}$$

$$= \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Matrix Representation of Composition & inverses

If $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ are linear transformations and B_U, B_V, B_W are bases for U, V, W respectively then

$$[T_2 \circ T_1]_{B_W, B_U} = [T_2]_{B_W, B_V} [T_1]_{B_V, B_U}$$

"cancellation"

$$\begin{tikzcd}
U \arrow[r, "T_1"] \arrow[d] \arrow[rr, bend left, "T_2 \circ T_1"] & V \arrow[r, "T_2"] \arrow[d] & W \arrow[d] \\
\mathbb{R}^n \arrow[r, "T_{A_1}"] & \mathbb{R}^m \arrow[r, "T_{A_2}"] & \mathbb{R}^k
\end{tikzcd}$$

where $A_1 = [T_1]_{B_V, B_U}$ where $A_2 = [T_2]_{B_W, B_V}$

$(T_{A_2} \circ T_{A_1})(x) = A_2(A_1 x) = (A_2 A_1)x$

So we get the formula above.

And if $T: V \to V$ is a linear operator and B is a basis for V then T is 1-1 (i.e. invertible) exactly when $[T]_B$ is invertible (as a matrix) and then $[T^{-1}]_B = [T]_B^{-1}$.

If we have the same basis B on both sides of the operator T we write $[T]_B$ instead of $[T]_{B,B}$.

<u>Another Example</u> The matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ -x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix}$.

i.e. $T_A(x) = Ax$ where $A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -1 \end{pmatrix}$ ← This is $[T]_{S_3, S_2}$

(This is not usual terminology!)

where $S_2 = \{e_1, e_2\}$, $S_3 = \{e_1, e_2, e_3\}$, the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$.

Now write the matrix for T_A with respect to the bases $B_2 = \{ \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{=u_1}, \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{=u_2} \}$ for $\mathbb{R}^2$ and

$B_3 = \{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{=v_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{=v_2}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{=v_3} \}$ for $\mathbb{R}^3$.

<u>Solution</u> Goal: Find $T_A(u_1)$, $T_A(u_2)$ in terms of v_1, v_2, v_3

$$T_A(u_1) = T_A\left(\begin{pmatrix}3\\1\end{pmatrix}\right) = \begin{pmatrix}9\\-1\\2\end{pmatrix} = 3\begin{pmatrix}1\\1\\0\end{pmatrix} + 6\begin{pmatrix}1\\0\\1\end{pmatrix} - 4\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$T_A(u_2) = T_A\left(\begin{pmatrix}0\\2\end{pmatrix}\right) = \begin{pmatrix}0\\4\\-2\end{pmatrix} = 3\begin{pmatrix}1\\1\\0\end{pmatrix} - 3\begin{pmatrix}1\\0\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

(To get the coefficients on right hand side, solve the appropriate linear system e.g. $\begin{pmatrix}9\\-1\\2\end{pmatrix} = av_1 + bv_2 + cv_3$.)

So in fact $[T_A(u_1)]_{B_3} = \begin{pmatrix}3\\6\\-4\end{pmatrix}$

and $[T_A(u_2)]_{B_3} = \begin{pmatrix}3\\-3\\1\end{pmatrix}$

and hence $[T_A]_{B_3,B_2} = \left([T_A(u_1)]_{B_3} \; [T_A(u_2)]_{B_3}\right) = \begin{pmatrix}3 & 3\\6 & -3\\-4 & 1\end{pmatrix}$.

<u>This example shows that, when looking for a matrix representation of a linear transformation $T: U \to V$, the choice of bases B_U, B_V for U and V matters!!!</u>

Very often (for computational reasons) we want to choose bases B_U, B_V to make $[T]_{B_V,B_U}$ as <u>simple</u> as possible.

First goal: understand how <u>changing</u> basis changes the matrix representation.

It's enough (for now) to work with linear operators $T : V \to V$. <u>Goal</u>: Choose basis B for V to make $[T]_B$ as simple as possible.

How does changing B change the matrix representation of T?

Change of Basis

Suppose we have an n-dimensional vector space V and two bases B, B'. We have a representation of any vector v in V in terms of ("old basis") B. We want to convert this to a representation of v in terms of ("new basis") B'.

In other words we want to encode the map $[v]_B \mapsto [v]_{B'}$ (for any v in V).

This map is just $I : V \longrightarrow V$ i.e. $I(v) = v$

domain has old basis B $= \{u_1, \ldots, u_n\}$

range has new basis B'

This is represented by the matrix

$$[I]_{B',B} = \left([I(u_1)]_{B'} \cdots [I(u_n)]_{B'} \right) = \left([u_1]_{B'} \cdots [u_n]_{B'} \right).$$

This is called the <u>transition matrix from B to B'</u>.

We have $[v]_{B'} = [I]_{B',B} [v]_B$.

<u>Example</u> In P_2, find the transition matrix from $B = \{1, x, x^2\}$ to $B' = \{1+x, 2x, x+x^2\}$, and the transition matrix from B' to B.

<u>Solution</u> Write $\overset{\text{old basis}}{\text{vectors}}$ in B in terms of $\overset{\text{new basis}}{\text{vectors}}$ in B'.

$$1 = (1+x) - \tfrac{1}{2}(2x)$$

$$x = \tfrac{1}{2}(2x)$$

$$x^2 = (x+x^2) - \tfrac{1}{2}(2x)$$

So $[I]_{B',B} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$

Now write old basis vectors in B' in terms of new basis vectors in B.

We can read this off & get

$$[I]_{B,B'} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ 1+x & 2x & x+x^2 \end{matrix} \end{array} \quad //$$

Notice $[I]_{B',B}\,[I]_{B,B'}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general $[I]_{B,B'} = [I]_{B',B}^{-1}$ i.e. the transition matrix from B' to B is the inverse of the transition matrix from B to B'!

Now we are all set to say how changing basis for V from B to B' changes the representation of a linear operator $T : V \to V$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\underset{\text{(with basis } B)}{V} \arrow[r, "{[T]_B}" above, "T" below] \arrow[d, shift right, "{[I]_{B',B}}\ I" left] & \underset{\text{(with basis } B)}{V} \arrow[d, "I\quad {[I]_{B',B}}"] \\
\underset{\text{(with basis } B')}{V} \arrow[r, "T" above, "?{[T]_{B'}}?" below] \arrow[u, shift right, "{[I]_{B',B}^{-1}}" right] & \underset{\text{(with basis } B')}{V}
\end{tikzcd}$$

So we have $\boxed{[T]_{B'} = [I]_{B',B} \cdot [T]_B \cdot [I]_{B',B}^{-1}}$

$$= [I]_{B,B'}^{-1}\,[T]_B\,[I]_{B,B'}$$

Example Find the matrix for $T: P_2 \to P_2$ given by

$$T(a+bx+cx^2) = 3a + (2a+b+c)x + 2cx^2$$

with respect to the basis $B' = \{1+x, 2x, x+x^2\}$.

Solution With respect to the standard basis $B = \{1, x, x^2\}$

T has matrix $[T]_B = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(either by hand or plug in each of the vectors from B into T in turn).

We already found $[I]_{B',B} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$

and $[I]_{B',B}^{-1} = [I]_{B,B'} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

So $[T]_{B'} = [I]_{B',B} [T]_B [I]_{B',B}^{-1} =$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad //$$

In other words

$T(1+x) = 3(1+x)$, $T(2x) = 2x$, $T(x+x^2) = 2(x+x^2)$.

So T scales the basis vectors of B'.

Computationally much simpler to work with

$[T]_{B'}$ (diagonal) than $[T]_B$.

Notice what the formula above (for $[T]_{B'}$ in terms of $[T]_B$) tells us about the relationship between $[T]_{B'}$ and $[T]_B$: they are <u>similar</u> matrices (there is some P with $[T]_{B'} = P^{-1}[T]_B P$ and in fact

$P = [I]^{-1}_{B',B} = [I]_{B,B'}$.

In fact: <u>Theorem</u> If $T: V \to V$ is a linear operator and B is a basis for V, and M is <u>any</u> matrix similar to $[T]_B$ (i.e. there is <u>some</u> matrix N with $M = N^{-1}[T]_B N$), then $M = [T]_{B'}$ for some basis B' for V and $N = [I]_{B,B'}$.

In other words <u>similar matrices represent the same linear operator (possibly with respect to different bases).</u>

So <u>goal</u> is now: given linear operator $T: V \to V$ and some matrix $[T]_B$ (i.e. w.r.t. B) we want to find "simplest" matrix M <u>similar</u> to $[T]_B$.

When can we diagonalize $[T]_B$?

When $[T]_B$ has n linearly independent eigenvectors $\leftarrow (c_1 \overset{\neq 0}{\cdots} c_n)^T$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) for which there is some λ in $\mathbb{R}$ (or $\mathbb{C}$) satisfying

$$[T]_B \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

(λ is called the corresponding eigenvalue)

Notice: $(c_1 \ldots c_n)^T$ are the coordinates of the vector $v = c_1 v_1 + \ldots + c_n v_n$ in V $(B = \{v_1, \ldots, v_n\})$

By definition of $[T]_B$, $T(v)$ is the vector in V with coordinates (w.r.t. B) given by $[T(v)]_B = [T]_B [v]_B$

$$= [T]_B \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

i.e. $T(v) = \lambda c_1 v_1 + \dots + \lambda c_n v_n$

$= \lambda (c_1 v_1 + \dots + c_n v_n) = \lambda v.$

Since similar matrices have the same eigenvalues, the choice of B above did not matter & it makes sense to define:

<u>Definition</u> If V is a real (resp. complex) vector space, then a vector v in V is an <u>eigenvector</u> for a linear operator $T: V \longrightarrow V$ if $v \neq 0$ and there is a real (resp. complex) number λ (the corresponding <u>eigenvalue</u>) with $T(v) = \lambda v$.

Given an eigenvalue λ, the span of all eigenvectors corresponding to λ is called the <u>eigenspace</u> of λ.

How to find eigenvalues/eigenvectors of $T: V \to V$?
The following are equivalent:

(1) λ is an eigenvalue of T;
(2) λ is an eigenvalue of $[T]_B$ for any basis B of V;
(3) The system of linear equations $(\lambda I - [T]_B)x = 0$ has a non-trivial solution;
(4) The operator $(\lambda I - T) : V \longrightarrow V$ $(v \mapsto \lambda v - T(v))$ has $\ker(\lambda I - T) \neq \{0\}$ i.e. $\lambda I - T$ is not invertible;

(5) λ is a solution to the characteristic equation for $[T]_B$ (for any B) : $\det(\lambda I - [T]_B) = 0$.

Can get analogue for linear operators:

Definition V finite dim. vector space
$T: V \to V$ linear operator

$\det(T) = \det([T]_B)$ for any basis B

(Determinants same for similar matrices.)
(as are characteristic equations)

Equivalent to list above is

(6) λ is a solution to characteristic equation for T $\det(\lambda I - T) = 0$.

3 remarks

1. We already saw $T: V \to V$ is invertible exactly when $[T]_B$ is invertible for some basis B — and hence for any basis B', since we now know that $[T]_B$ and $[T]_{B'}$ are similar & "invertibility" is preserved under similarity.

2. Remember that a square matrix eg. $[T]_B$ is invertible exactly when its determinant is non-zero, i.e. $\lambda = 0$ is NOT an eigenvalue (see (5)). By 1. and the equivalence of (1) and (2) above, we have that T is itself invertible exactly when $\lambda = 0$ is NOT an eigenvalue of T.

3. By these equivalences, we also see that T is invertible exactly when $\det(T) \neq 0$.

To find a diagonal representation of T, as said in class, we need to see if some (any) representation $[T]_B$ can be diagonalized.

i.e. Step 1 Find $[T]_B$ for some basis B

Step 2 Find the eigenvalues $\lambda_1, \dots, \lambda_k$ of $[T]_B$

Step 3 Find the corresponding eigenvectors $w_1, \dots, w_k$. Are they linearly independent? Is $k = n$? If yes to both, then:

T can be represented by $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \left[= \begin{pmatrix} w_1 \cdots w_n \end{pmatrix}^{-1} [T]_B \begin{pmatrix} w_1 \cdots w_n \end{pmatrix} \right]$.