Name ______________________________

Student Number ______________________________

MATH 2R03

SAMPLE SOLUTIONS

SUMMER SEMESTER 2018
DURATION OF EXAM: 2.5 Hours
MCMASTER UNIVERSITY FINAL EXAM (PRACTICE)

Dr. Margaret E. M. Thomas
EVENING CLASS
August 2, 2018

THIS TEST PAPER INCLUDES **12** PAGES AND **7** QUESTIONS, PLUS A BONUS QUESTION (#8). IT IS PRINTED ON BOTH SIDES OF THE PAPER. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF AN INVIGILATOR.

- Please fill in your name and student number above. You may do this before the test starts.
- Do not open the test paper until the test begins! Once the test starts, please fill in your initials and student number where indicated at the top of each subsequent page.
- Attempt all questions.
- The total number of available points is **45**. Points are indicated next to each question.
- You may use a standard McMaster calculator, Casio FX-991 MS or MS Plus (no communication capability); no other aids are permitted.
- Write your answers in the corresponding spaces provided on the test paper.
- You must show your work to get full credit.
- Pages 10–12 at the end are provided for rough work; please ask an invigilator for more rough paper if needed. Please write your student number and initials clearly at the top of each extra page used, and hand in all paper along with your test paper.

Good Luck.

Score

Question	1	2	3	4	5	6	7	8 (Bonus)
Points	10	5	5	5	5	5	5	5
Score								

1. (10 points) For each of the following statements, state if it is true or false, and provide a BRIEF justification for your answer (explanation or counterexample).

   (a) In a complex inner product space V, $\langle u, v\rangle \neq \overline{\langle u, v\rangle}$, for every u, v in V.

   F

   This says "$\langle u,v\rangle$ never equals $\overline{\langle u,v\rangle}$" but consider $u=3, v=6$, $\langle\cdot,\cdot\rangle$ the complex dot product on $\mathbb{C}$; then $\langle u,v\rangle = 3\cdot 6 = 18$
   $\overline{\langle u,v\rangle} = \overline{3\cdot 6} = \overline{18} = 18 \; \}=.$

   (b) If A in $M_n(\mathbb{R})$ is an orthogonal matrix, then $||Ax|| = 1$ for all x in $\mathbb{R}^n$.

   F

   e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (orthogonal: $A^T = A^{-1}$)
   & $x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$; $\|Ax\| = \left\|\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\| = \sqrt{13} \neq 1.$

   Any counter-example acceptable!

   (c) If B, B' are two bases of a vector space V, then the transition matrix $[I]_{B',B}$ is invertible.

   T

   $[I]_{B',B}^{-1} = [I]_{B,B'}$

   (d) Every Hermitian matrix is normal.

   T

   If A is Hermitian then $A = \overline{A}^T (= A^*)$
   So $AA^* = A\overline{A}^T = A^2$ and $A^*A = \overline{A}^T A = A^2$ so $AA^* = A^*A$
   i.e. A normal.

   (e) If $T\colon U \to V$ is a linear transformation between finite dimensional vector spaces with $\dim(U) = \dim(V)$, then T is an isomorphism.

   F

   e.g. $T: M_2(\mathbb{R}) \longrightarrow \mathbb{R}^4$
   $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a,0,0,0)$ is NOT an isomorphism, but $\dim M_2(\mathbb{R}) = \dim \mathbb{R}^4 = 4.$
   (NOT 1-1 OR onto !!)
   (Any counterexample acceptable!)

2. (5 points) For each part, place the answer on the line provided.

(a) Find the eigenvalues of $T\colon U \to U$ if, for a basis B of U, $[T]_B = \begin{pmatrix} 3-i & 2i \\ 0 & -4+i \end{pmatrix}$.

$3-i,\ -4+i$

(The eigenvalues of a triangular matrix are the diagonal entries)

(b) Write down an integral expression for the Fourier coefficient of $\sin(5x)$ within the Fourier series for $f(x) = e^x$. on $[0,2\pi]$ TYPO.

$$\frac{1}{\pi}\int_0^{2\pi} e^x \sin(5x)\,dx.$$

(c) If $T\colon U \to V$ is a linear transformation between finite-dimensional vector spaces with $\dim(U) = 5$ and the dimension of the kernel of T is equal to 3, then what is the smallest $\dim(V)$ could be?

2

(Rank – Nullity Theorem says

$\dim(U) = \dim(\ker(T)) + \dim(R(T))$

So $5 = 3 + \underbrace{\dim R(T)}_{\leq \dim(V)}$ So $\dim(V) \geq 2$.

(d) Write the quadratic form $x^2 - 9y^2 + 25z^2 - 4xy - 6xz + 5yz$ in matrix notation.

$$X^T \begin{pmatrix} 1 & -2 & -3 \\ -2 & -9 & 5/2 \\ -3 & 5/2 & 25 \end{pmatrix} X.$$

(e) Write down the inverse of the matrix $\begin{pmatrix} \frac{1}{\sqrt{3}}i & \frac{\sqrt{2}}{\sqrt{3}}i \\ -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$.

$$\begin{pmatrix} -1/\sqrt{3}\, i & -\sqrt{2}/\sqrt{3} \\ -\sqrt{2}/\sqrt{3}\, i & 1/\sqrt{3} \end{pmatrix}$$

This matrix A is unitary: it has orthonormal rows & orthonormal columns. So $A^{-1} = \bar{A}^T$.

3. (5 points) Diagonalize the quadratic form

$$q(x,y,z) = -2x^2 + 2y^2 + 2z^2 - 4yz$$

and say if it is positive definite, negative definite or indefinite.

Write $q(x,y,z) = X^T \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}}_{A} X$ & diagonalize A.

Solve $0 = \det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & 0 & 0 \\ 0 & 2-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{pmatrix} = (-2-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix}$

$= (-2-\lambda)\left((2-\lambda)^2 - 4\right) = -(2+\lambda)(\lambda^2 - 4\lambda) = -\lambda(\lambda+2)(\lambda-4)$.

So A has eigenvalues $\lambda = 0$, $\lambda = -2$, $\lambda = 4$.

Find eigenspace basis for each eigenvalue:

$\lambda = 0$ Solve $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} X = 0 \xrightarrow[\text{Elim.}]{\text{Gauss.}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ So $x_1 = 0$ and x_3 free and $x_2 = x_3$ i.e. basis is eg. $\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

$\lambda = -2$ Solve $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} X = 0 \xrightarrow[\text{Elim.}]{\text{Gauss.}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ So $x_2 = 0 = x_3$ and x_1 free So basis is e.g. $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.

$\lambda = 4$ Solve $\begin{pmatrix} -6 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} X = 0 \xrightarrow[\text{Elim.}]{\text{Gauss}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ So $x_1 = 0$ and $x_2 = -x_3$ so basis is e.g. $\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$ (or $\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$)

We need an orthonormal set of eigenvectors to diagonalize a quadratic form so take $\left\{\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\right\}$ i.e. if $Py = X$ then $y^T D y = X^T A X$ where $D = P^T A P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ where $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$

There is at least one positive eigenvalue and at least one negative eigenvalue so $q(x,y,z)$ is indefinite.

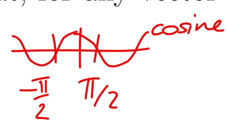
4. (5 points) Let V be an inner product space.

   (a) Write down the formula for the angle between two vectors u and v in V.

$\uparrow \theta$

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} \qquad \left(\text{or } \theta = \cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\cdot\|v\|}\right)\right)$$

   (b) Let W be a subspace of V. Show that, for any vector u in V, if θ is the angle between u and $\text{proj}_W u$, then θ lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



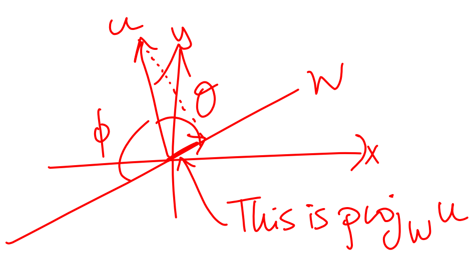
We want to show that $\cos\theta \geq 0$. Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis for W.

By above formula, $\cos\theta = \dfrac{\langle u, \text{proj}_W u\rangle}{\|u\|\cdot\|\text{proj}_W u\|}$

$$= \frac{1}{\|u\|\cdot\|\text{proj}_W u\|}\left\langle u, \frac{\langle u, v_1\rangle}{\|v_1\|^2}v_1 + \cdots \frac{\langle u, v_n\rangle}{\|v_n\|^2}v_n\right\rangle$$

$$= \frac{1}{\|u\|\cdot\|\text{proj}_W u\|}\left(\frac{\langle u, v_1\rangle^2}{\|v_1\|^2} + \cdots + \frac{\langle u, v_n\rangle^2}{\|v_n\|^2}\right) \quad \leftarrow \text{Everything in this expression is} \geq 0.$$

   (c) Give a brief geometric description in $\mathbb{R}^2$ of the fact shown in (b).



The angle between u and $\text{proj}_W u$ is θ & not ϕ

($\text{proj}_W u$ always closest vector in W to u)

5. (5 points) Find a unitary matrix P that unitarily diagonalizes A, and determine P^*AP, where

$$A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}.$$

Find eigenvalues:

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} i-\lambda & 1 \\ -1 & i-\lambda \end{pmatrix} = (i-\lambda)^2 + 1 = \lambda^2 - 2i\lambda = \lambda(\lambda - 2i)$$

So the eigenvalues are $\lambda = 0, 2i$.

Find eigenvectors:

$\underline{\lambda = 0}$ $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \to \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$ So $x_1 = ix_2$. Basis is e.g. $\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$

$\underline{\lambda = 2i}$ $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \to \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ So $x_1 = -ix_2$. Basis is e.g. $\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

Find orthonormal eigenvectors:

$$\left\|\begin{pmatrix} i \\ 1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$ So get $\left\{\begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$.

Then $P = \begin{pmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ and $P^{-1}AP = P^*AP = \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}$.

6. (5 points) Consider linear transformations $T_1\colon U \to V$ and $T_2\colon V \to W$ between vector spaces U, V and W. Suppose that T_1 is onto and that the composition, $T_2 \circ T_1\colon U \to W$, is $1-1$. Show that T_2 is $1-1$.

To show that T_2 is 1-1, we take 2 vectors v_1, v_2 in V and assume $T_2(v_1) = T_2(v_2)$. We want to show $v_1 = v_2$.

Since T_1 is onto V and v_1, v_2 lie in V, there are vectors u_1, u_2 in U with $T_1(u_1) = v_1$, $T_1(u_2) = v_2$.

Then $T_2(v_1) = T_2(v_2)$ can be written as

$$T_2(T_1(u_1)) = T_2(T_1(u_2))$$

i.e. $(T_2 \circ T_1)(u_1) = (T_2 \circ T_1)(u_2)$.

Since $T_2 \circ T_1$ is 1-1, we have that $u_1 = u_2$.

But then T_1 is a well-defined function so $T_1(u_1) = T_1(u_2)$ i.e. $v_1 = v_2$, as required.

7. (5 points)

(a) Suppose that V is a subspace of the vector space of real-valued, differentiable functions on $\mathbb{R}$ and that $B = \{1, \cos(2x), \sin(2x)\}$ is a basis for V. Consider the linear transformation $T : V \to V$ defined by $T(f(x)) = 2f(x) - f'(x)$, where $f'(x)$ is the first derivative of $f(x)$. Find the matrix $[T]_B$.

in terms of B:

Find $T(1) = 2.1 - 0 = 2$

$T(\cos(2x)) = 2\cos(2x) + 2\sin(2x) = 2\cos(2x) + 2\sin(2x)$

$T(\sin(2x)) = 2\sin(2x) - 2\cos(2x) = -2\cos(2x) + 2\sin(2x)$

basis vectors of B

So $[T]_B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & 2 \end{pmatrix}$

(b) Use the matrix $[T]_B$ found in part (a) to compute $T(5 + 3\cos(2x) - \sin(2x))$ (show working).

$[5 + 3\cos(2x) - \sin(2x)]_B = (5, 3, -1)$

So compute $[T]_B \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 4 \end{pmatrix}$

This is $[T(5 + 3\cos(2x) - \sin(2x))]_B$

i.e. $T(5 + 3\cos(2x) - \sin(2x)) = 10 + 8\cos(2x) + 4\sin(2x)$.

8. (Bonus; 5 points) Consider the vector space of all real-valued functions on $\mathbb{R}$. For each $t \in \mathbb{R}$, define the function

$$f_t(x) := \begin{cases} 0 & x < t \\ 1 & x \geq t \end{cases} \qquad x \text{ in } \mathbb{R}.$$

Show that the set of all such functions $\{f_t : t \text{ in } \mathbb{R}\}$ is linearly independent.

We need to show that whenever we take a finite linear combination of functions in the set i.e. $c_1 f_{t_1} + c_2 f_{t_2} + \dots + c_n f_{t_n}$ for t_i all different from one another, then if this linear combination equals 0, we must have $c_i = 0$ for each $i = 1, \dots, n$.

So suppose that $c_1 f_{t_1} + \dots + c_n f_{t_n} = 0$ (the zero function).

Suppose without loss that $t_1 < t_2 < \dots < t_n$. (If not, rename the numbers that they represent so that this is true.)

Take x_1 in (t_1, t_2). Then $f_{t_1}(x_1) = 1$ (as $x > t_1$)

but $f_{t_2}(x_1) = f_{t_3}(x_1) = \dots = f_{t_n}(x_1) = 0$

(as $x_1 < t_2 < t_3 < \dots < t_n$)

So

$$\underset{\uparrow}{0} = (c_1 f_{t_1} + \dots + c_n f_{t_n})(x_1) = c_1 f_{t_1}(x_1) + \dots + c_n f_{t_n}(x_1) = c_1 \text{ by above.}$$

(By assumption above)

So now we have $c_2 f_{t_2} + \dots + c_n f_{t_n} = 0$.

Continue, taking x_2 in (t_2, t_3) $\dashrightarrow$ get $c_2 = 0$, and so on.

Repeating this argument, we end up with $c_i = 0$ for all i, as required.

ROUGH WORK

ROUGH WORK

ROUGH WORK

THE END