Name ______________________________

Student Number ______________________________

MATH 2R03

SUMMER SEMESTER 2018 — Dr. Margaret E. M. Thomas
DURATION OF EXAM: 2.5 Hours
MCMASTER UNIVERSITY FINAL EXAM (PRACTICE) — August 2, 2018

THIS TEST PAPER INCLUDES **12** PAGES AND **7** QUESTIONS, PLUS A BONUS QUESTION. IT IS PRINTED ON BOTH SIDES OF THE PAPER. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF AN INVIGILATOR.

- Please fill in your name and student number above. You may do this before the test starts.
- Do not open the test paper until the test begins! Once the test starts, please fill in your initials and student number where indicated at the top of each subsequent page.
- Attempt all questions.
- The total number of available points is **45**. Points are indicated next to each question.
- You may use a standard McMaster calculator, Casio FX-991, MS or MS Plus (no communication capability); no other aids are permitted.
- Write your answers in the corresponding spaces provided on the test paper.
- You must show your work to get full credit.
- Three sides at the end are provided for rough work; please ask an invigilator for more rough paper if needed. Please write your student number and initials clearly at the top of each extra page used, and hand in all paper along with your test paper.

Good Luck.

Score

Question	1	2	3	4	5	6	7	8
Points	10	5	5	5	5	5	5	5
Score								

1. (10 points) For each of the following statements, state if it is true or false, and provide a BRIEF justification for your answer (explanation or counterexample).

   (a) In a complex inner product space V, $\langle u, v\rangle \neq \overline{\langle u, v\rangle}$, for every u, v in V.

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   (b) If A in $M_n(\mathbb{R})$ is an orthogonal matrix, then $||Ax|| = 1$ for all x in $\mathbb{R}^n$.

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   (c) If B, B' are two bases of a vector space V, then the transition matrix $[I]_{B',B}$ is invertible.

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   (d) Every Hermitian matrix is normal.

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   (e) If $T\colon U \to V$ is a linear transformation between finite dimensional vector spaces with $\dim(U) = \dim(V)$, then T is an isomorphism.

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2. (5 points) For each part, place the answer on the line provided.

   (a) Find the eigenvalues of $T\colon U \to U$ if, for a basis B of U, $[T]_B = \begin{pmatrix} 3-i & 2i \\ 0 & -4+i \end{pmatrix}$.

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   (b) Write down an integral expression for the Fourier coefficient of $\sin(5x)$ within the Fourier series for $f(x) = e^x$.

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   (c) If $T\colon U \to V$ is a linear transformation between finite-dimensional vector spaces with $\dim(U) = 5$ and the dimension of the kernel of T is equal to 3, then what is the smallest $\dim(V)$ could be?

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   (d) Write the quadratic form $x^2 - 9y^2 + 25z^2 - 4xy - 6xz + 5yz$ in matrix notation.

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   (e) Write down the inverse of the matrix $\begin{pmatrix} \frac{1}{\sqrt{3}}i & \frac{\sqrt{2}}{\sqrt{3}}i \\ -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$.

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3. (5 points) Diagonalize the quadratic form

$$q(x, y, z) = -2x^2 + 2y^2 + 2z^2 - 4yz$$

and say if it is positive definite, negative definite or indefinite.

4. (5 points) Let V be an inner product space.

   (a) Write down the formula for the angle between two vectors u and v in V.

   (b) Let W be a subspace of V. Show that, for any vector u in V, if θ is the angle between u and $\text{proj}_W u$, then θ lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

   (c) Give a brief geometric description in $\mathbb{R}^2$ of the fact shown in (b).

5. (5 points) Find a unitary matrix P that unitarily diagonalizes A, and determine P^*AP, where

$$A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}.$$

6. (5 points) Consider linear transformations $T_1 \colon U \to V$ and $T_2 \colon V \to W$ between vector spaces U, V and W. Suppose that T_1 is onto and that the composition, $T_2 \circ T_1 \colon U \to W$, is $1-1$. Show that T_2 is $1-1$.

7. (5 points)

   (a) Suppose that V is a subspace of the vector space of real-valued, differentiable functions on $\mathbb{R}$ and that $B = \{1, \cos(2x), \sin(2x)\}$ is a basis for V. Consider the linear transformation $T : V \to V$ defined by $T(f(x)) = 2f(x) - f'(x)$, where $f'(x)$ is the first derivative of $f(x)$. Find the matrix $[T]_B$.

   (b) Use the matrix $[T]_B$ found in part (a) to compute $T(5 + 3\cos(2x) - \sin(2x))$ (show working).

8. (Bonus; 5 points) Consider the vector space of all real-valued functions on $\mathbb{R}$. For each $t \in \mathbb{R}$, define the function

$$f_t(x) := \begin{cases} 0 & x < t \\ 1 & x \geq t \end{cases} \qquad x \text{ in } \mathbb{R}.$$

Show that the set of all such functions $\{f_t : t \text{ in } \mathbb{R}\}$ is linearly independent.

ROUGH WORK

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THE END