Name ______________________________

Student Number ______________________________

MATH 2R03 – PRACTICE MIDTERM 1

SOLUTIONS

SUMMER SEMESTER 2018 Dr. Margaret E. M. Thomas
DURATION OF MIDTERM: 1 Hour

THIS TEST PAPER INCLUDES **8** PAGES AND **6** QUESTIONS. IT IS PRINTED ON BOTH SIDES OF THE PAPER. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF AN INVIGILATOR.

- Please fill in your name and student number above. You may do this before the test starts.
- Do not open the test paper until the test begins! Once the test starts, please fill in your initials and student number where indicated at the top of each subsequent page.
- Attempt all questions.
- The total number of available points is **30**. Points are indicated next to each question.
- You may use a standard McMaster calculator, Casio FX-991, MS or MS Plus (no communication capability); no other aids are permitted.
- Write your answers in the corresponding spaces provided on the test paper.
- You must show your work to get full credit.
- One side at the end is provided for rough work; please ask an invigilator for more rough paper if needed. Please write your student number and initials clearly at the top of each extra page used, and hand in all paper along with your test paper.

Good Luck.

Score

Question	1	2	3	4	5	6	**Total**
Points	5	3	7	5	5	5	30
Score							

1. (5 points in total – 1 point for each)
For each of the following statements, declare (without further explanation) if it is true or false.

(a) For all complex numbers z, it holds that $\bar{z} = iz$.

FALSE (If $z = a+bi$, then $\bar{z} = a - bi \neq iz = -b + ai$ if $a \neq -b$
e.g. $z = 3+i$, $3 \neq -1$.)

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(b) In a vector space, if $\{v_1, \ldots, v_k\}$ is a linearly independent set, then $\{v_1, \ldots, v_k\} \setminus \{v_k\}$ is linearly dependent.

FALSE (e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a linearly independent set in $\mathbb{R}^3$, but so is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.)

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(c) In a real inner product space, $\langle u, v \rangle > 0$ for all distinct vectors u, v.

FALSE (e.g. in $\mathbb{R}^2$, $\begin{pmatrix} -3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -3 - 4 = -7$.)

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(d) The set $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis for $M_{22}(\mathbb{R})$.

FALSE ($M_{22}(\mathbb{R})$ has dimension 4 so a basis needs 4 vectors.)

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(e) In any inner product space, $||u||$ is real, for every vector u.

TRUE ($||u|| = \sqrt{\langle u, u \rangle}$ ← This is the positive square root of the real number $\langle u, u \rangle$.)

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2. (3 points in total – 1 point for each)
For each part, circle the number corresponding to the ONE possible correct way to complete the sentence.

(a) A set of vectors $S = \{v_1, \ldots, v_k\}$ in a vector space V is a basis for V if

(1) there are infinitely many ways to write every vector in V as a linear combination of the vectors in S;

(2) S contains the zero vector and is closed under addition and multiplication;

(3) S is linearly independent and S spans V.

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(b) When two complex numbers in polar form are multiplied together, the modulus and argument of the product are respectively obtained by

(1) adding their moduli together and multiplying their arguments together;

(2) multiplying their moduli together and multiplying their arguments together;

(3) multiplying their moduli together and adding their arguments together;

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(c) The distance between two vectors u and v in an inner product space is given by

(1) $d(u, v) = u - v$;

(2) $d(u, v) = \sqrt{\langle u - v, u - v\rangle}$; $= ||u-v||$

(3) $d(u, v) = ||u|| - ||v||$.

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3. (7 points in total)

(a) (2 points) If $z_1 = 3 + i$, $z_2 = 6 - 2i$, find $\overline{z_1 \bar{z}_2}$.

$$\overline{z_1 \bar{z}_2} = \overline{(3+i)(6+2i)} = \overline{18+6i+6i+2i^2}$$
$$= \overline{16+12i}$$
$$= 16-12i$$

or $\overline{z_1\bar{z}_2} = \bar{z}_1 z_2 = (3-i)(6-2i) = 18-6i-6i+2i^2$
$= 16-12i.$

(b) (3 points) Find all complex numbers z satisfying $z^3 = -1$.

Put -1 in polar form: $-1 = e^{i\pi}$.

Then z satisfying $z^3=-1$ have modulus $\sqrt[3]{1} = 1$

and arguments $\frac{\pi + 2\pi k}{3}$ for $k=0,1,2$ i.e. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

So the complex numbers z with $z^3=-1$ are $z = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$

or $z = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}), \cos(\pi) + i\sin(\pi), \cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3})$

or any other equivalent form.

(c) (2 points) If $u = (1 + i, -3i)$, $v = (3 - 2i, 5 + i)$ are vectors in $\mathbb{C}^2$, find $u \cdot v$.

$$u \cdot v = (1+i)\overline{(3-2i)} + (-3i)\overline{(5+i)}$$
$$= (1+i)(3+2i) + (-3i)(5-i)$$
$$= \cancel{3}+3i+2i-2-15i-\cancel{3}$$
$$= \underline{\underline{-2-10i}}$$

4. (5 points)

The following defines a real inner product on $\mathbb{R}^3$: set $\langle u, v\rangle = Au \cdot Av$, where A is the matrix $\begin{pmatrix} 2 & 1 & 0 \\ -3 & 2 & 5 \\ 1 & -1 & -4 \end{pmatrix}$, and $\cdot$ is the usual dot product in $\mathbb{R}^3$.

Find the norm of the vector $u = (6, 1, -2)^T$.

$$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{Au \cdot Au}.$$

Find Au: $\begin{pmatrix} 2 & 1 & 0 \\ -3 & 2 & 5 \\ 1 & -1 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 13 \\ -26 \\ 13 \end{pmatrix}$

Then $\|u\| = \sqrt{Au \cdot Au} = \sqrt{\begin{pmatrix} 13 \\ -26 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -26 \\ 13 \end{pmatrix}}$

$$= 13\sqrt{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

$$= 13\sqrt{1 + 4 + 1}$$

$$= \underline{\underline{13\sqrt{6}}}.$$

5. (5 points)

Show that if W is the set of all polynomials of degree at most 4 which satisfy $p(0) = 0$, then W is a subspace of P_4 (the vector space of all polynomials of degree at most 4).

In order to show that W is a subspace, we need to show that (i) W is not empty;
(ii) W is closed under addition;
(iii) W is closed under scalar multiplication.

To show (i), notice that the zero polynomial is 0 at 0, so $\vec{0}$ is in W. Thus W is not empty.

To show (ii), let p and q be polynomials in W i.e. $p(0) = q(0) = 0$.

Then $(p+q)(0) = p(0) + q(0) = 0$, so $p+q$ is in W. Thus W is closed under addition.

To show (iii), let p be in W and let a be a scalar.

Then $(ap)(0) = a \cdot p(0) = 0$, so ap is in W. Thus W is closed under scalar multiplication.

Hence W is a subspace of P_2.

6. (5 points in total)

(a) (3 points)
Show that if u and v are orthogonal, then $||u+v|| = ||u-v||$.

If $\langle u,v \rangle$ are orthogonal, then $\langle u,v\rangle = 0$.

$$||u+v||^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle$$
$$= ||u||^2 + 2\cancel{\langle u,v\rangle} + ||v||^2$$
$= 0$ by orthogonality
$$= ||u||^2 + ||v||^2$$

$$||u-v||^2 = \langle u-v, u-v\rangle = \langle u,u\rangle - \langle v,u\rangle - \langle u,v\rangle + \langle v,v\rangle$$
$$= ||u||^2 - 2\cancel{\langle u,v\rangle} + ||v||^2$$
$= 0$ by orthogonality

So $||u+v||^2 = ||u-v||^2$, so $||u+v|| = ||u-v||$.

(b) (2 points)
Suppose that, in a real inner product space, u and v are non-zero vectors that are orthogonal to one another. Show that $\{u, v\}$ is a linearly independent set. (*Hint*: for all vectors w, $\langle w, 0\rangle = 0$.)

Let $0 = au + bv$. We want to show that $a = b = 0$.

$$0 = \langle u, 0\rangle = \langle u, au+bv\rangle$$
$$= a\langle u,u\rangle + b\cancel{\langle u,v\rangle}$$
$\to 0$ by orthogonality

i.e. $a||u||^2 = 0$. Since $u \neq 0$, $||u||^2 \neq 0$, so we must have $a = 0$.

Similarly $0 = \langle v, 0\rangle = \langle v, au+bv\rangle$
$$= a\cancel{\langle v,u\rangle} + b\langle v,v\rangle$$
0 by orthogonality

i.e. $b||v||^2 = 0$. Since $v \neq 0$, $||v||^2 \neq 0$, so we must have $b = 0$.

ROUGH WORK

THE END