Name ______________________________

Student Number ______________________________

MATH 2R03 – PRACTICE MIDTERM 2

SAMPLE SOLUTIONS

SUMMER SEMESTER 2018 — Dr. Margaret E. M. Thomas

DURATION OF MIDTERM: 1 Hour

THIS TEST PAPER INCLUDES **8** PAGES AND **5** QUESTIONS. IT IS PRINTED ON BOTH SIDES OF THE PAPER. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF AN INVIGILATOR.

- Please fill in your name and student number above. You may do this before the test starts.
- Do not open the test paper until the test begins! Once the test starts, please fill in your initials and student number where indicated at the top of each subsequent page.
- Attempt all questions.
- The total number of available points is **30**. Points are indicated next to each question.
- You may use a standard McMaster calculator, Casio FX-991, MS or MS Plus (no communication capability); no other aids are permitted.
- Write your answers in the corresponding spaces provided on the test paper.
- You must show your work to get full credit.
- Two sides at the end are provided for rough work; please ask an invigilator for more rough paper if needed. Please write your student number and initials clearly at the top of each extra page used, and hand in all paper along with your test paper.

Good Luck.

Score

Question	1	2	3	4	5	**Total**
Points	10	6	5	4	5	30
Score						

1. (10 points in total – 2 points for each)
For each of the following statements, state if it is true or false, and provide a VERY BRIEF justification (explanation or counterexample).

(a) If A is an invertible matrix, then A has a QR decomposition.

T

If A is invertible, then A has linearly independent column vectors, so A has a QR decomposition.

(b) The transformation $T: M_2(\mathbb{R}) \to: M_2(\mathbb{R})$ given by $T(A) = A - A^T$ is linear.

T

$T(A+B) = (A+B) - (A+B)^T = (A - A^T) + (B - B^T) = T(A) + T(B)$
$T(kA) = kA - (kA)^T = kA - kA^T = k(A - A^T) = kT(A)$.
So T is linear.

(c) The space P_5 of real polynomials of degree at most 5 is isomorphic to $M_{3\times 2}(\mathbb{R})$.

T

Both P_5 and $M_{3\times 2}(\mathbb{R})$ have dimension 6.

(d) If $T: U \to V$ is a linear transformation with nullity$(T) = 2$, dim $(U) = 3$ and dim $(V) = 4$, then rank$(T) = 2$.

F

$\dim(U) = \text{nullity}(T) + \text{rank}(T)$ (by the Rank Nullity Theorem)
$3 = 2 + \text{rank}(T)$
So $\text{rank}(T) = 1 \neq 2$.

(e) In an inner product space V with dim $(V) = n$, any orthogonal set of n vectors is a basis for V.

~~T~~ F

Any orthogonal set of non-zero vectors is linearly independent, and any linearly independent set of n vectors in an n-dimensional space V spans V, so the set is a basis.

The question was supposed to say "any orthogonal set of n non-zero vectors is a basis." This is true, for the reason given. However, as the question actually stands, the statement is false, as 0 could be one of those vectors & hence the set would not be linearly independent, hence not a basis.

2. (6 points in total – 2 points for each)
For each part, circle ALL the number(s) corresponding to possible correct way(s) to complete each sentence.

Note: You are NOT asked to write anything here in order to justify your answers. These notes are just to help explain the more mysterious answers.

(a) In a QR decomposition of a matrix A,
   (1) R is invertible;
   (2) Q is square;
   (3) R is lower triangular;
   (4) R satisfies $R^T R = 0$;
   (5) Q has orthogonal column vectors.

(orthonormal = orthogonal to each other + unit vectors)

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(b) For a subspace W of a vector space V and a vector u in V,
   (1) $\langle \text{proj}_W u, u\rangle = 0$;
   (2) $||\text{proj}_W u|| = 1$;
   (3) $\langle \text{proj}_{W^\perp} u, u\rangle = 0$;
   (4) $||u - w|| \geq ||u - \text{proj}_W u||$ for any vector w in W;
   (5) $\langle \text{proj}_W u, \text{proj}_{W^\perp} u\rangle = 0$.

u does not necessarily lie in either W or $W^\perp$ so is not necessarily orthogonal to either $\text{proj}_W u$ or $\text{proj}_{W^\perp} u$.

$||u-w|| > ||u-\text{proj}_W u||$ for any vector w in W other than $\text{proj}_W u$ (and obviously setting $w = \text{proj}_W u$ gives $||u-w|| = ||u-\text{proj}_W u||$)

lies in W lies in $W^\perp$

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(c) If a linear transformation $T: U \to V$ is an isomorphism, then
   (1) $\ker(T) = \{0\}$; this is another way of saying T is 1-1
   (2) U = V;
   (3) $T(x) = T(y)$ implies $x = y$, for vectors x and y in U; this is another way of saying T is 1-1
   (4) $\dim(U) = \dim(V)$;
   (5) $R(T) = V$. another way of saying T is onto

= 1-1 + onto

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3. (5 points) In the space P_3 of real polynomials of degree at most 3 with the inner product given by $\langle a_0+a_1x+a_2x^2+a_3x^3, b_0+b_1x+b_2x^2+b_3x^3\rangle = a_0b_0+a_1b_1+a_2b_2+a_3b_3$, use the Gram–Schmidt Process to find an orthonormal basis for the subspace spanned by $\{\underbrace{x^2-1}_{u_1}, \underbrace{x+2}_{u_2}, \underbrace{x^3-\sqrt{3}x}_{u_3}\}$.

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{x^2-1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}x^2 - \frac{1}{\sqrt{2}}.$$

Find $u_2 - \langle u_2, v_1\rangle v_1 = x+2 - \langle x+2, \frac{1}{\sqrt{2}}x^2 - \frac{1}{\sqrt{2}}\rangle\left(\frac{1}{\sqrt{2}}x^2-\frac{1}{\sqrt{2}}\right)$

$$= x+2+\sqrt{2}\left(\frac{1}{\sqrt{2}}x^2-\frac{1}{\sqrt{2}}\right) = x+2+x^2-1$$

$$= x^2+x+1.$$

Then $v_2 = \dfrac{x^2+x+1}{\|x^2+x+1\|} = \dfrac{x^2+x+1}{\sqrt{1^2+1^2+1^2}} = \dfrac{1}{\sqrt{3}}x^2+\dfrac{1}{\sqrt{3}}x+\dfrac{1}{\sqrt{3}}.$

Find $u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2 =$

$$x^3-\sqrt{3}x - \cancel{\langle x^3-\sqrt{3}x, \tfrac{1}{\sqrt{2}}x^2-\tfrac{1}{\sqrt{2}}\rangle}^{0}\left(\tfrac{1}{\sqrt{2}}x^2-\tfrac{1}{\sqrt{2}}\right)$$

$$- \langle x^3-\sqrt{3}x, \tfrac{1}{\sqrt{3}}x^2+\tfrac{1}{\sqrt{3}}x+\tfrac{1}{\sqrt{3}}\rangle\left(\tfrac{1}{\sqrt{3}}x^2+\tfrac{1}{\sqrt{3}}x+\tfrac{1}{\sqrt{3}}\right)$$

$$= x^3-\sqrt{3}x + \left(\frac{1}{\sqrt{3}}x^2+\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right) = x^3+\frac{1}{\sqrt{3}}x^2+\underbrace{\left(-\sqrt{3}+\frac{1}{\sqrt{3}}\right)}_{\frac{-3+1}{\sqrt{3}}}x+\frac{1}{\sqrt{3}}$$

$$= x^3+\frac{1}{\sqrt{3}}x^2-\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}.$$

$$v_3 = \frac{x^3+\frac{1}{\sqrt{3}}x^2-\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}}{\|x^3+\frac{1}{\sqrt{3}}x^2-\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\|} = \frac{x^3+\frac{1}{\sqrt{3}}x^2-\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}}{\sqrt{1^2+(\frac{1}{\sqrt{3}})^2+(\frac{2}{\sqrt{3}})^2+(\frac{1}{\sqrt{3}})^2}} = \frac{x^3+\frac{1}{\sqrt{3}}x^2-\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}}{\sqrt{1+\frac{1}{3}+\frac{4}{3}+\frac{1}{3}}}$$

$$= \frac{1}{\sqrt{3}}x^3+\frac{1}{3}x^2-\frac{2}{3}x+\frac{1}{3}$$

4. (4 points) In the space $M_{2\times 3}(\mathbb{R})$ with the standard inner product $\langle A, B\rangle = \mathrm{tr}(B^T A)$, write

$$C = \begin{pmatrix} -5 & -4 & 2 \\ 1 & -3 & 0 \end{pmatrix}$$

as $C = D + E$, where D lies in $W = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 0 & 3 & -5 \\ 4 & 2 & 0 \end{pmatrix}}_{u_2} \right\}$ and E lies in $W^\perp$.

First we need an orthogonal basis for W.

(Note: do not need to normalize as only need an orthogonal, not an orthonormal, basis.)

Take $v_1 = u_1$ and

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$\begin{pmatrix} 0 & 3 & -5 \\ 4 & 2 & 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 & 3 & -5 \\ 4 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & -5 \\ 4 & 2 & 0 \end{pmatrix} - \frac{\overset{-12}{\cancel{(-5)2 + 2(-1)}}}{\underset{6}{\cancel{1^2 + 2^2 + (-1)^2}}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -5 \\ 4 & 2 & 0 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}$$

We write $C = \overbrace{\mathrm{proj}_W C}^{D} + \overbrace{\mathrm{proj}_{W^\perp} C}^{E}$.

Then $D = \mathrm{proj}_W C = \dfrac{\langle C, v_1\rangle}{\|v_1\|^2} v_1 + \dfrac{\langle C, v_2\rangle}{\|v_2\|^2} v_2$

$$= \frac{\left\langle \begin{pmatrix} -5 & -4 & 2 \\ 1 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle}{6} v_1 + \frac{\left\langle \begin{pmatrix} -5 & -4 & 2 \\ 1 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix} \right\rangle} v_2$$

$$= \frac{\overset{2}{\cancel{(-5)1 + 2\cdot 2 + (-3)(-1)}}}{6} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} + \frac{\overset{-20}{\cancel{(-5)\cdot 2 + (-4)\cdot 3 + 2(-1) + 1\cdot 4}}}{\underset{30}{\cancel{2^2 + 3^2 + (-1)^2 + 4^2}}} \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & -1/3 & 0 \end{pmatrix} - \begin{pmatrix} 4/3 & 2 & -2/3 \\ 8/3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 4/3 \\ -8/3 & -1/3 & 0 \end{pmatrix}$$

And $E = \mathrm{proj}_{W^\perp} C = C - \mathrm{proj}_W C = \begin{pmatrix} -5 & -4 & 2 \\ 1 & -3 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -2 & 4/3 \\ -8/3 & -1/3 & 0 \end{pmatrix} = \begin{pmatrix} -4 & -2 & 2/3 \\ 11/3 & -8/3 & 0 \end{pmatrix}$.

5. (5 points) If U, V and W are vector spaces such that U is isomorphic to V, and V is isomorphic to W, show (just using definitions) that U is isomorphic to W.

If U is isomorphic to V, then there is an isomorphism $T_1 : U \to V$ and if V is isomorphic to W, then there is an isomorphism $T_2 : V \to W$.

We want to show $T_2 \circ T_1 : U \to W$ is an isomorphism i.e. is 1-1 and onto.

1-1 If x and y are vectors in V with $(T_2 \circ T_1)(x) = (T_2 \circ T_1)(y)$

i.e. $T_2(T_1(x)) = T_2(T_1(y))$, then

T_2 1-1 $\Rightarrow T_1(x) = T_1(y)$ and then T_1 1-1 $\Rightarrow x = y$.

So $T_2 \circ T_1$ is 1-1.

Onto If w is a vector in W, then, since T_2 is onto, there is a vector v in V with $T_2(v) = w$.

Since T_1 is onto, there is a vector u in U with $T_1(u) = v$.

Then $(T_2 \circ T_1)(u) = T_2(T_1(u)) = T_2(v) = w$. So $T_2 \circ T_1$ is onto.

$\square$

(Note: In particular, you are not told U and V and W are finite dimensional so you cannot argue with finite bases.)

ROUGH WORK

ROUGH WORK

THE END