3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 15

Today **TWO RANDOM VARIABLES**

- Two (or conceivably more) random variables i.e. measurements of interest at same time
- may or may not be related (& relationship may not be obvious)

We'll look only at continuous random variables.

Definition The joint probability density function for 2 continuous r.v.s X and Y is denoted $f_{XY}(x,y)$ and satisfies

(1) $f_{X,Y}(x,y) \geq 0$

(2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$ (Volume under graph of f_{XY})

(3) $P((X,Y) \in R) = \iint_R f_{XY}(x,y)\,dx\,dy$

for any region R in xy-plane.

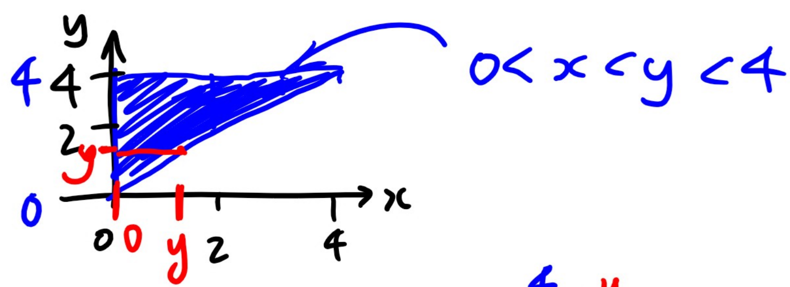
<u>Example</u> Suppose X and Y are continuous random variables with $f_{XY}(x,y) = c(x+y)$ with $0 < X < Y < 4$.

(a) Find c

(b) $P(X < 3, Y < 2)$

(c) $P(X < 2, Y < 3)$.

<u>Solution</u> First identify domain of f_{XY}:

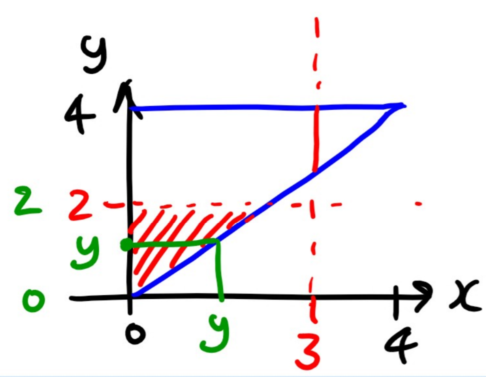


$$1 = \iint_{\text{domain}} f_{XY}(x,y)\,dx\,dy = \int_0^4 \int_0^y c(x+y)\,dx\,dy$$

$$= \int_0^4 \left[\frac{cx^2}{2} + cyx\right]_0^y dy$$

$$= \int_0^4 \underbrace{\frac{cy^2}{2} + cy^2}_{\frac{3cy^2}{2}}\,dy = \left[\frac{cy^3}{2}\right]_0^4 = \underline{32c}.$$

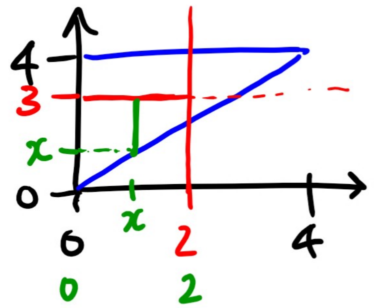
$\Rightarrow$ $c = 1/32$.



(b) $P(X < 3, Y < 2)$

$$= \int_0^2 \int_0^y \frac{1}{32}(x+y)\, dx\, dy = \dots = \left[\frac{y^3}{64}\right]_0^2 = \frac{8}{64} = \underline{\underline{\frac{1}{8}}}.$$

(c) $P(X<2,\ Y<3)$



$$\int_0^2 \int_x^3 \frac{1}{32}(x+y)\, dy\, dx$$

$$= \int_0^2 \left[\frac{xy}{32} + \frac{y^2}{64}\right]_x^3 dx$$

$$= \int_0^2 \frac{3x}{32} + \frac{9}{64} - \frac{x^2}{32} - \frac{x^2}{64}\, dx$$

$$\dots = \left[\frac{32x^2 + 9x - x^3}{64}\right]_0^2 = \frac{11}{32}$$

5.2 Marginal Probability Distributions

If we have X and Y, continuous r.v.s with joint pdf f_{XY}, each of X and Y has its own pdf & own prob. distribution referred to in <u>this</u> context as marginal pdf / prob distribution.

called f_X and f_Y respectively.

We have $f_X(x) = \int f_{XY}(x,y)\, dy$

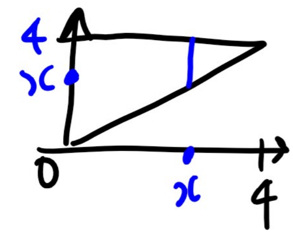
$$f_Y(y) = \int f_{XY}(x,y)\,dx$$

So in the example above with $0 < X < Y < 4$ and

$$f_{XY}(x,y) = \frac{1}{32}(x+y)$$

$$f_X(x) = \int_x^4 \frac{1}{32}(x+y)\,dy$$

$$= \cdots = -\frac{3x^2}{64} + \frac{x}{8} + \frac{1}{4}$$



$$f_Y(y) = \int_0^y \frac{1}{32}(x+y)\,dx = \frac{3y^2}{64}.$$

So now e.g. $P(X < x) = \int_{-\infty}^{x} f_X(t)\,dt$

& sim. for Y and $f_Y(y)$.

& we can see that the mean μ_X of X is

$$E(X) = \int x f_X(x)\,dx$$

and μ_Y and σ_X^2 and σ_Y^2 similarly.

<u>5.4 Independence</u> X and Y are <u>independent</u>

if • events involving only X are independent of events involving only Y (E, F indep. $P(E\cap F) = P(E)P(F)$)

- $P(X\in A, Y\in B) = P(X\in A)P(Y\in B)$.
- If X, Y are jointly distributed with joint pdf f_{XY} then $f_{XY}(x,y) = f_X(x) f_Y(y)$

<u>Notice</u> In earlier example $f_{XY}(x,y) = \frac{1}{32}(x+y)$ for $0 < X < Y < 4$, X, Y <u>not</u> independent (knowledge of value of one of the r.v.s changes possible value of other).

In general X, Y independent $\Rightarrow$ domain f_{XY} is rectangular

Alas $\Leftarrow$ not necessarily true (i.e. rectangular domain does not guarantee independence)

<u>Example</u> If $f_{XY}(x,y) = 2x+y$ $\quad 0 < x < 1$, $\; -\frac{1}{2} < y < \frac{1}{2}$

are X and Y independent?

<u>Solution</u> We need to check $f_{XY}(x,y) \overset{?}{=} f_X(x) f_Y(y)$

Find f_X & f_Y :

$$f_X(x) = \int_{-1/2}^{1/2} 2x+y \; dy = \dots = 2x$$

$$f_Y(y) = \int_0^1 2x+y \; dx = \dots = 1+y$$

$$f_X(x) f_Y(y) = 2x(1+y) = 2x + 2xy$$

$$\neq 2x+y$$

X,Y NOT independent.