3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 26

Last time HYPOTHESIS TESTING

H_0 : Equation about a parameter e.g. $\mu = 50$

H_1 : Contradictory statement e.g. $\mu \neq 50, \mu > 50, \mu < 50$

If value of test statistic falls in critical region, then reject H_0 & accept H_1; if not, then fail to reject H_0.

Type I Error : reject H_0 (for H_1) when H_0 true (prob. α)

Type II Error : fail to reject H_0 when H_1 true (prob. β)

Default : H_0

To decide based on data evidence to reject H_0 (& accept H_1 instead): strong conclusion.

To decide to stick with H_0 i.e. fail to reject H_0 because we didn't see enough data evidence to reject H_0 is a weak conclusion.

The power of a test is the probability of correctly rejecting H_0 in favour of H_1

$= P(\text{reject } H_0 \text{ when } H_1 \text{ true}) = 1 - \beta.$

So bigger power = better.
(smaller β)

P-value   When we carry out our test,
we get a value for test statistic.

Based on that we reject H_0 (for H_1) or
fail to reject H_0,
based on some predetermined significance level α

Since this cannot distinguish between test stat. value slightly in critical region or way into it, we define:

The p-value is probability of obtaining a value of the test statistic at least as extreme as the observed value. ↳ in the direction of H_1.

In other words, p-value is the smallest significance level α that would lead to rejection of H_0 given the data.

Example   Yesterday our example was Normal pop.
$\sigma = 10$, $n = 35$, and $H_0 : \mu = 50$
$H_1 : \mu \neq 50$

If observed value of $\bar{x} = 54$, what is the p-value?

Solution Identify the values as extreme as $\bar{x} = 54$:

at least

≥ 54 , ≤ 46 (4 away from 50)

So p-value $P(\bar{X} \geq 54 \text{ or } \bar{X} \leq 46)$

$$= P\left(Z \geq \frac{54-50}{10/\sqrt{35}}\right) + P\left(Z \leq \frac{46-50}{10/\sqrt{35}}\right)$$

$$= 2P(Z \geq 2.37) \quad \text{by symmetry}$$

$$= 2(1 - 0.991106) = \underline{0.018}.$$

Remark The smaller the p-value the stronger the evidence for rejecting H_0 for H_1.

We can have an α significance level to determine if we should reject H_0 for H_1 (or not) but now can report p-value as well.

i.e. p-value $\leq \alpha$ then reject H_0 for H_1

p-value $> \alpha$ then fail to reject H_0.

In the setup of previous example if we had

$H_0 : \mu = 50$, $H_1 : \mu > 50$

and observation $\bar{x} = 54$

then p-value $= P(\bar{X} \geq 54)$

as we only reject H_0 for H_1 if $\bar{x}$ is too big.

If $H_0 : \mu = 50$, $H_1 : \mu < 50$ and observed value of $\bar{x} = 47$, then p-value $= P(\bar{X} \leq 47)$

(as only reject H_0 if $\bar{x}$ too small)

<u>Remark</u> Close relationship between Hypothesis Testing & Confidence Intervals:

If we have

$$H_0 : \theta = \theta_0 \leftarrow \text{a \# } \theta_0 \text{, called "null value"}$$
$$H_1 : \theta \neq \theta_0$$

then a test with significance level α will reject H_0 in favour of H_1 if observed value $\hat{\theta}$ lies <u>outside</u> a $100(1-\alpha)\%$ confidence interval for θ.

- C.I. provides range of likely values for θ using the data at a stated confidence level ($100(1-\alpha)\%$)

- Hypothesis Testing provides risk value (p-value) for some decision at a stated significance level α.

9.2 Hypothesis Tests on Mean of Normal Distribution, Variance Known

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$ (so $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$)

$H_0 : \mu = \mu_0 \leftarrow$ some #

$H_1 : \mu \neq \mu_0$, $\mu > \mu_0$, $\mu < \mu_0$
(Case I) (Case II) (Case III)

What is the test statistic?

Earlier we always chose $\bar{X}$.

Since we always standardize, assuming H_0,

let's choose $Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$ from the outset as the test statistic

$\sim N(0,1)$

<u>IF</u> H_0 true.

So to test μ, with σ^2 known, at significance level α :

<u>Step 1</u> Compute $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$ from data.

Step 2 Reject H_0 if p-value $\leq \alpha$.

Case I $H_1 : \mu \neq \mu_0$. What is p-value?

(Note, the calculation done in class was unfortunately messy. It's written out more fully below.)

P-value if $\bar{x} > \mu_0$

$$= \underbrace{P(Z_0 \geqslant z_0)} + \underbrace{P(Z_0 \leq -z_0)} \leq \alpha$$

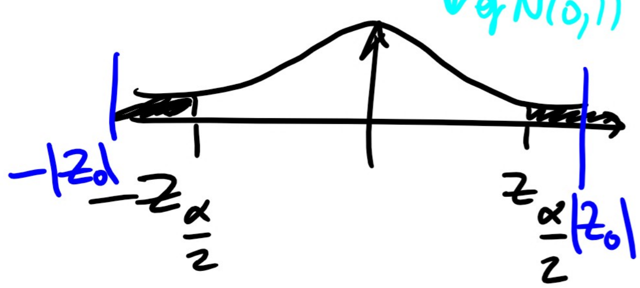
(If $\bar{x} < \mu_0$, then p-value by symmetry is almost same. — negative of the terms — see below)

By symmetry the prob are same. of $N(0,1)$

i.e. $P(Z \geqslant z_0) \leq \frac{\alpha}{2}$

Happens if $|z_0| > z_{\frac{\alpha}{2}}$

or $-|z_0| < -z_{\frac{\alpha}{2}}$.



i.e. reject H_0 for H_1 if $|z_0| > z_{\frac{\alpha}{2}}$ or $|z_0| < -z_{\frac{\alpha}{2}}$

The book lists this situation as simply $z_0 > z_{\alpha/2}$

or $z_0 < -z_{\alpha/2}$

which I object to for the reasons alluded to in class, so I changed what I wrote here at the end to include $|\cdot|$ signs. In practice it will be clear from the #s produced for z_0 and $-z_0$ which is bigger than μ_0 and which is smaller than μ_0, and so you won't get mixed up!

But to spell it out:

If $\bar{x} > \mu_0$, then one extreme value is $\geqslant \bar{x}$

and the analogous extreme value on the other side is $\mu_0 - (\bar{x} - \mu_0)$.

e.g. in Example earlier, $\bar{x} = 54$, $\mu_0 = 50$, so one extreme value was 54 and the other extreme value was

$$46 = 50 - (54 - 50) = \mu_0 - (\bar{x} - \mu_0)$$

Then when we look at the p-value

we get $$P(\bar{X} \leq \mu_0 - (\bar{x} - \mu_0)) + P(\bar{X} \geqslant \bar{x})$$

$$= P\left(Z_0 \leq \frac{\mu_0 - (\bar{x} - \mu_0) - \mu_0}{\sigma/\sqrt{n}}\right) + P\left(Z_0 \geqslant \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$= P\Bigg(Z_0 \leq \underbrace{\frac{-(\bar{x} - \mu_0)}{\sigma/\sqrt{n}}}_{-z_0}\Bigg) + P\Bigg(Z_0 \geqslant \underbrace{\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}}_{z_0}\Bigg).$$

But if the observed value $\bar{x} < \mu_0$, then "at least as extreme as $\bar{x}$" means either $\leq \bar{x}$ or $\geqslant \mu_0 + (\mu_0 - \bar{x})$, and hence the p-value is given by $P(\bar{X} \leq \bar{x}) + P(\bar{X} \geqslant \mu_0 + (\mu_0 - \bar{x}))$

$$= P\left(Z_0 \leq \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right) + P\left(Z_0 \geqslant \frac{\mu + (\mu_0 - \bar{x}) - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$= P\left(Z_0 \leq \underbrace{\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}}_{z_0}\right) + P\left(Z_0 \geq \underbrace{\frac{-(\bar{x}-\mu_0)}{\sigma/\sqrt{n}}}_{-z_0}\right)$$

So if $\bar{x} > \mu_0$ we get p-value $= P(Z_0 \leq -z_0) + P(Z_0 \geq z_0)$ and $z_0 > 0$

and if $\bar{x} < \mu_0$ we get p-value $= P(Z_0 \leq z_0) + P(Z_0 \geq -z_0)$ and $z_0 < 0$.

That means that, regardless of whether $\bar{x} > \mu_0$ or $\bar{x} < \mu_0$, we get p-value $= P(Z_0 \leq -|z_0|) + P(Z_0 \geq |z_0|)$.

Hence we reject H_0 in favour of H_1 when p-value $\geq \alpha$

i.e. when $\underbrace{P(Z_0 \leq -|z_0|)} + \underbrace{P(Z_0 \geq |z_0|)} \geq \alpha$

Since both of these probabilities are equal by the symmetry about the origin of the $N(0,1)$ distribution, we have that we should reject H_0 in favour of H_1 when $P(Z_0 \geq |z_0|) \geq \frac{\alpha}{2}$.

When does z_0 satisfy this inequality?

When $|z_0|$ is MORE extreme than $z_{\alpha/2}$, the value that satisfies $P(Z_0 \geq z_{\alpha/2}) = \alpha/2$.

So we reject H_0 if $|z_0| > z_{\alpha/2}$ or $|z_0| < -z_{\alpha/2}$.