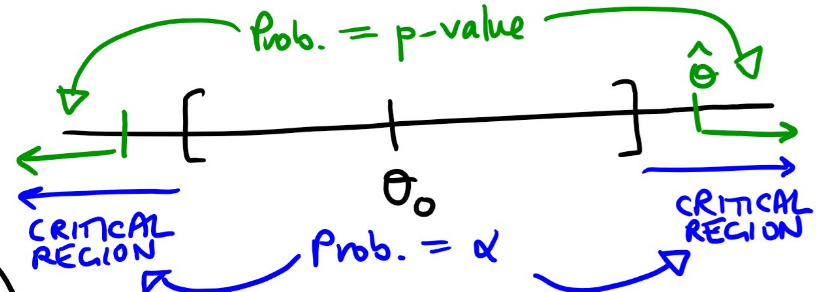
3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 27

Last time P-VALUE



Reject H_0 if p-value $< \alpha$.

TEST ON MEAN OF NORMAL, VARIANCE KNOWN $N(\mu, \sigma^2)$

$H_0: \mu = \mu_0$

H_1: (I) $\mu \neq \mu_0$ or (II) $\mu > \mu_0$ or (III) $\mu < \mu_0$

Use test stat. $Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$.

Case I p-value: $2 \cdot P\left(Z_0 > \left|\dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right|\right)$

Reject H_0 if $\left|\dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}$ or...

$-\left|\dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| < -z_{\alpha/2}$

if $|z_0| > z_{\alpha/2}$ or $-|z_0| < -z_{\alpha/2}$

Case II

Reject H_0 for H_1 when $z_0 > z_\alpha$

Case III

Reject H_0 for H_1 when $z_0 < -z_\alpha$.

Since reference is $N(0,1)$ the above is called a **z - test**.

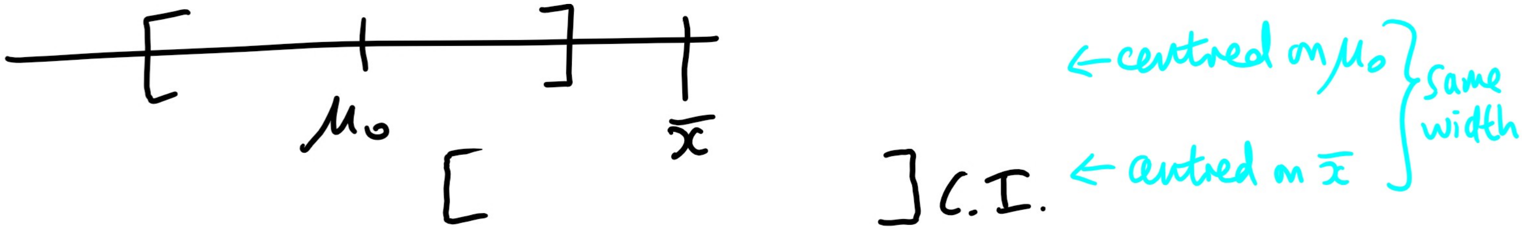
If we unwrap above, we get thats critical region(s) closely resemble(s) the $100(1-\alpha)\%$ C.I. bounds but relative to μ_0 not $\bar{x}$:

I) $H_1 : \mu \neq \mu_0$: CR $\bar{x} < \mu_0 - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$

or $\bar{x} > \mu_0 + z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$

II) $H_1 : \mu > \mu_0$: CR $\bar{x} > \mu_0 + z_{\alpha}\left(\frac{\sigma}{\sqrt{n}}\right)$

III) $H_1 : \mu < \mu_0$: CR $\bar{x} < \mu_0 - z_{\alpha}\left(\frac{\sigma}{\sqrt{n}}\right)$



Example A sample of 25 adults gave mean body temp. of 36.8°C. Body temp. assumed to be Normal with $\sigma = 0.34$°C.

Test $H_0 : \mu = 37$°C
$H_1 : \mu \neq 37$°C at level $\alpha = 0.01$.

Solution $$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{36.8 - 37}{0.34/5} = -2.94$$

P-value : $P(Z_0 \leq -2.94) + P(Z_0 \geq 2.94)$

$= 2(0.001641) = 0.003282.$

$< \alpha = 0.01.$

In this case, reject H_0 (mean is unlikely to be 37°C).

<u>Recall</u> $\beta = P(\text{Type II error})$

We fix $\alpha = P(\text{Type I error})$ & this affects β
(α & β in opposition)
but we can still influence β by changing sample size n.

β depends on true mean $\mu = \mu_0 + \delta$, say.
(so $\mu_0 = \mu - \delta$)

So how does β depend on δ.

In 2-sided case:

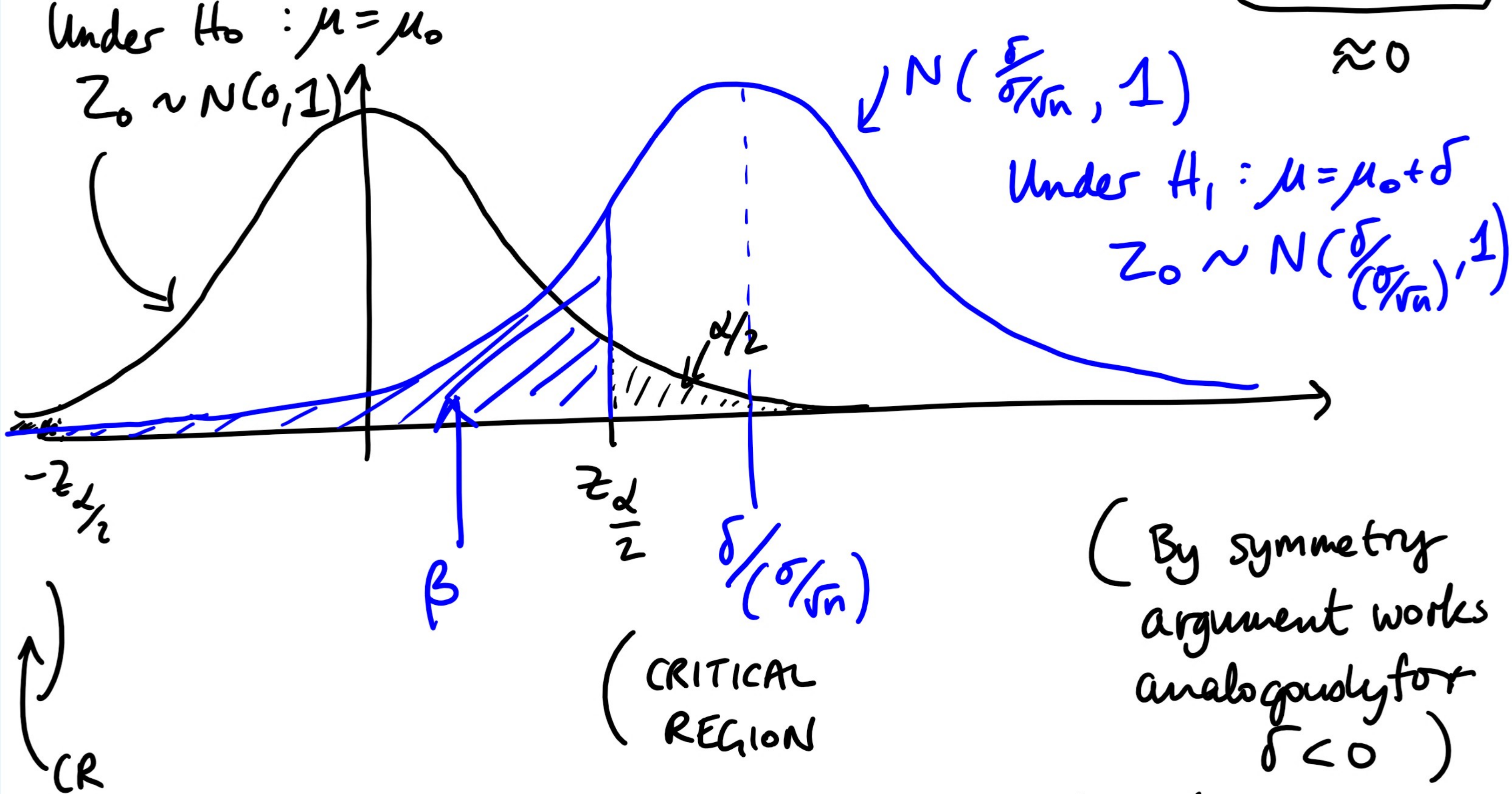
$$\beta(\delta) = P\left(-z_{\frac{\alpha}{2}} < Z_0 < z_{\frac{\alpha}{2}}\right)$$

when $$Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{X} - (\mu - \delta)}{\sigma/\sqrt{n}}$$

$$= \underbrace{\underbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}_{\sim N(0,1)} + \frac{\delta}{(\sigma/\sqrt{n})}}_{\sim N\left(\frac{\delta}{\sigma/\sqrt{n}},\ 1\right)}$$

$$\beta(\delta) = P\left(-z_{\alpha/2} - \frac{\delta}{\sigma/\sqrt{n}} < Z < z_{\alpha/2} - \delta/\sigma/\sqrt{n}\right)$$

(standardizing) $\qquad = \Phi\left(z_{\frac{\alpha}{2}} - \frac{\delta}{(\sigma/\sqrt{n})}\right) - \underbrace{\Phi\left(-z_{\frac{\alpha}{2}} - \frac{\delta}{(\sigma/\sqrt{n})}\right)}_{\approx 0}$



(By symmetry argument works analogously for $\delta < 0$)
& get an analogue for 1-sided situation (replace $z_{\frac{\alpha}{2}}$ with z_α as appropriate)

From above we have

$$z_{\alpha/2} - \frac{\delta}{\sigma/\sqrt{n}} \approx -z_\beta$$

So if we fix β & we know $\mu = \mu_0 + \delta$

we find n :

$$\left(\frac{z_{\frac{\alpha}{2}} + z_\beta}{\delta}\right)^2 \sigma^2 \approx n$$

<u>Example</u> Battery life in hours $\sim N(\mu, 1.5)$

10 batteries have sample mean 40.5 hours. If the true mean life is 43 hours, how big a sample do

we need to ensure $\beta < 0.05$ in a 2-sided test with $H_0 : \mu = 40$, $H_1 : \mu \neq 40$ at $\alpha = 0.025$?

<u>Solution</u>

$$n \approx \left(\frac{z_{\alpha/2} + z_\beta}{\delta}\right)^2 \sigma^2$$

$\underset{\shortparallel}{\mu_1}$ $\underset{\shortparallel}{\mu_0}$

$\delta = 43 - 40$
$= 3$

$$= \left(\frac{z_{0.0125} + z_{0.05}}{3}\right)^2 (1.5)$$

For

$\left.\begin{array}{l} z_{0.0125} = 2.24 \\ z_{0.05} = 1.64 \end{array}\right\} \rightarrow$

$$= \left(\frac{2.24 + 1.64}{3}\right)^2 (1.5) = 2.51$$

So $n = 3$ enough.

9.3 Hypothesis Test on Normal Mean, Variance UNknown

Analogous to C.I. setting —
Use t-distribution.

Here analogous test statistic is

$$T_0 = \frac{\bar{X} - \mu_0}{S / \sqrt{n}}$$

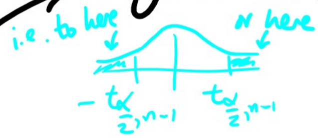
which has a t-distribution with $n-1$ degrees of freedom

All setup completely analogous to known variance setting: ← but with t_{n-1} distribution instead of $N(0,1)$ distribution.

$$\frac{\bar{x}-\mu_0}{s/\sqrt{n}}$$

$H_0 : \mu = \mu_0$

H_1 : (I) $\mu \neq \mu_0$ → Reject H_0 if $|t_0| > t_{\frac{\alpha}{2}, n-1}$ or $-|t_0| < -t_{\frac{\alpha}{2}, n-1}$



(II) $\mu > \mu_0$ → Reject H_0 if $t_0 > t_{\alpha, n-1}$

(III) $\mu < \mu_0$ → Reject H_0 if $t_0 < -t_{\alpha, n-1}$.